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A note on knot concordance and involutive knot Floer homology

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Abstract. We prove that if two knots are concordant, then their involutive knot Floer complexes satisfy a certain type of stable equivalence.

1. Introduction

The knot Floer homology package of Ozsváth-Szabó [**OS04**] and Rasmussen [**Ras03**] has many applications to concordance. For example, many different smooth concordance invariants can be extracted from the filtered chain homotopy type of the knot Floer complex, such as τ [**OS03**], $\Upsilon(t)$ [**OSS17**], and ν^+ [**HW16**]. Furthermore, the second author [**Hom14**] showed that, modulo an appropriate equivalence relation, the set of knot Floer complexes forms a group, and that there is a homomorphism from the knot concordance group to this group. In [**Hom17**, Theorem 1], she showed that if two knots are concordant, then their knot Floer complexes satisfy a certain type of stable equivalence.

Recently, Manolescu and the first author [**HM17a**] used the conjugation symmetry on Heegaard Floer complexes to define involutive Heegaard Floer homology. They similarly considered the conjugation action on the knot Floer complex. Zemke [**Zem17**] showed that, under an appropriate equivalence relation, the set of knot Floer complexes together with the extra structure given by the conjugation action form a group, and that there is a homomorphism from the knot concordance group to this group. The aim of this note is to prove an involutive analog of [**Hom17**, Theorem 1]. Throughout, $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

Theorem 1. *If K is slice, then $(CFK^\infty(K), \iota_K)$ is filtered chain homotopic to $(\mathbb{F}[U, U^{-1}], \mathrm{id}) \oplus (A, \iota_A)$, where A is acyclic, i.e., $H_*(A) = 0$.*

Corollary 2. *If K_1 and K_2 are concordant, then we have the following filtered chain homotopy equivalence*

$$(CFK^\infty(K_1), \iota_{K_1}) \oplus (A_1, \iota_{A_1}) \simeq (CFK^\infty(K_2), \iota_{K_2}) \oplus (A_2, \iota_{A_2}),$$

where A_1, A_2 are acyclic, i.e., $H_*(A_1) = H_*(A_2) = 0$.

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2. Background

In 2013, Manolescu introduced a $Pin(2)$-equivariant version of Seiberg-Witten Floer homology and used it to resolve the Triangulation Conjecture [**Man16**]. Since then, several authors have given applications of this invariant, especially to the homology cobordism group [**Man14, Lin15b, Sto15b, Sto15a, Sto16**]. F. Lin also gave a reformulation to monopole Floer homology, and deduced various applications [**Lin14, Lin15a, Lin16c, Lin16b, Lin16a**].

Two years later, Manolescu and the first author introduced a shadow of $Pin(2)$-equivariant Seiberg-Witten Floer homology, called *involutive Heegaard Floer homology* [**HM17b**], in Ozsváth-Szabó's Heegaard Floer homology [**OSz04**]. Involutive Heegaard Floer homology has had a number of applications, again mainly to the homology cobordism group [**HMZ17, BH16, DM17, Zem17, HL17**].

Like ordinary Heegaard Floer homology, involutive Heegaard Floer homology has a version for knots: Manolescu and the first author associate to a knot K an order-four symmetry ι_K on the knot Floer complex $CFK^\infty(K)$, and extract various concordance invariants from this data [**HM17b**]. In [**Zem17**], Zemke studies the behavior of these complexes and the associated involutions under connected sum. In this section, we recap some of his definitions and results, in preparation for proving Theorem 1 in Section 3. We begin with the following definition, which is a specialization of [**Zem17**, Definition 2.2].

Definition 2.1. We say that $(C, \partial, B, \iota_C)$ is an *ι_K-complex* if

- (C, ∂) is a finitely-generated, free, $\mathbb{Z}$-graded, $(\mathbb{Z}\oplus\mathbb{Z})$-filtered, $\mathbb{F}[U, U^{-1}]$-complex with a filtered basis B;
- Given an element $x \in B$, $\partial x = \sum_{y\in B} U^{n_y} y$ for some set of integers $n_y \geq 0$;
- The action of U lowers homological grading by 2 and each filtration level by 1;
- There is an isomorphism $H_*(C, \partial) \cong \mathbb{F}[U, U^{-1}]$;
- ι_C is a skew-filtered U-equivariant endomorphism of C;
- $\iota_C^2 \simeq \mathrm{id} + \Phi_B \circ \Psi_B$, where $\Phi_B \colon C \to C$ and $\Psi_B \colon C \to C$ are formal derivatives of ∂.

(For more on the definition of the maps Φ_B and Ψ_B, see [**Zem17**, p. 7].)

Typically we omit the differential and basis from the notation. This definition is not quite Zemke's; one can think of our ι_K-complexes as the part of his ι_K-complexes concentrated in Alexander grading zero [**Zem17**, Remark 2.3]. If K is a knot, then $(CFK^\infty(K), \iota_K)$ can be made into an ι_K-complex by picking a basis for $CFK^\infty(K)$. The following notion of equivalence between two ι_K-complexes is particularly useful for studying concordance.

Definition 2.2. [**Zem17**, Definition 2.4] Two ι_K-complexes (C_1, ι_{C_1}) and (C_2, ι_{C_2}) are said to be *locally equivalent* if there are filtered, grading-preserving $\mathbb{F}[U, U^{-1}]$-equivariant chain maps

$$F\colon C_1 \to C_2 \qquad G\colon C_2 \to C_1$$

such that F and G induce isomorphisms on homology and furthermore

$$F \circ \iota_{C_1} \simeq \iota_{C_2} \circ F \qquad G \circ \iota_{C_2} \simeq \iota_{C_1} \circ G$$

via skew-filtered U-equivariant chain homotopy equivalences. (If in addition $F \circ G \simeq \mathrm{id}$ and $G \circ F \simeq \mathrm{id}$ via filtered U-equivariant chain homotopy equivalences, the ι_K-complexes are said to be *homotopy equivalent.*)

One can define two possible products on the set of ι_K-complexes, denoted $\times_1$ and $\times_2$, and given by

$$(C_1, \iota_{C_1}) \times_1 (C_2, \iota_{C_2}) = (C_1 \otimes C_2, \iota_{C_1} \otimes \iota_{C_2} + (\Phi_{B_1} \otimes \Psi_{B_2}) \circ (\iota_{C_1} \otimes \iota_{C_2}))$$
$$(C_1, \iota_{C_1}) \times_2 (C_2, \iota_{C_2}) = (C_1 \otimes C_2, \iota_{C_1} \otimes \iota_{C_2} + (\Psi_{B_1} \otimes \Phi_{B_2}) \circ (\iota_{C_1} \otimes \iota_{C_2}))$$

Zemke shows that $(C_1, \iota_{C_1}) \times_1 (C_2, \iota_{C_2})$ is filtered chain-homotopy equivalent to $(C_1, \iota_{C_1}) \times_2 (C_2, \iota_{C_2})$. Following similar work in [**Sto15b**] and [**HMZ17**], Zemke further shows that either of these products makes the set of ι_K-complexes up to the relationship of local equivalence into an abelian group $\mathfrak{I}_K$ [**Zem17**, Proposition 2.6]. One then obtains a homomorphism from $\mathcal{C}$ the smooth knot concordance group to $\mathfrak{I}_K$ as follows.

Proposition 2.3. [**Zem17**, Theorem 1.5] *Let $\mathcal{C}$ be the smooth knot concordance group. The map*

$$\mathcal{C} \to \mathfrak{I}_K$$
$$K \mapsto [CFK^\infty(K), \iota_K]$$

is a well-defined group homomorphism.

Zemke [**Zem17**, Definition 2.5 and Proposition 2.6] shows that the inverse of $[C, \iota_C]$ is $[C^*, \iota_C^*]$, where $C^* = \mathrm{Hom}_{\mathbb{F}[U,U^{-1}]}(C, \mathbb{F}[U, U^{-1}])$, the map ι^* is the dual of ι, and B^* is a dual basis to B. The identity element of $\mathfrak{I}_K$ is $[\mathbb{F}[U, U^{-1}], \mathrm{id}]$.

3. Proof of Theorem

Since Zemke [**Zem17**, Theorem 1.5] showed that concordant knots have locally equivalent ι_K-complexes, Theorem 1 and Corollary 2 follow immediately from the following proposition and corollary.

Proposition 3.1. *If (C, ι_C) is locally equivalent $(\mathbb{F}[U, U^{-1}], \mathrm{id})$, then (C, ι_C) is filtered chain homotopy equivalent to*

$$(\mathbb{F}[U, U^{-1}], \mathrm{id}) \oplus (A, \iota_A),$$

where A is some acyclic complex, i.e., $H_*(A) = 0$.

Here (A, ι_A) is not technically an ι_K complex, because $H_*(A)$ is trivial rather than $\mathbb{F}[U, U^{-1}]$. The notation should be taken to indicate that (A, ι_A) satisfies all other structural requirements in the definition of an ι_K-complex.

Corollary 3.2. *Two ι_K-complexes (C_1, ι_{C_1}) and (C_2, ι_{C_2}) are locally equivalent if and only if there is a filtered chain homotopy equivalence*

$$(C_1, \iota_{C_1}) \oplus (A_1, \iota_{A_1}) \simeq (C_2, \iota_{C_2}) \oplus (A_2, \iota_{A_2}),$$

for some acyclic complexes A_1 and A_2.

Proof of Corollary 3.2. The reverse direction is straightfoward: $(C_1, \iota_{C_1}) \oplus (A_1, \iota_{A_1})$ is locally equivalent to (C_1, ι_{C_1}), so if $(C_1, \iota_{C_1}) \oplus (A_1, \iota_{A_1})$ is filtered homotopy equivalent to $(C_2, \iota_{C_2}) \oplus (A_2, \iota_{A_2})$, or even merely locally equivalent, then (C_1, ι_{C_1}) is locally equivalent to (C_2, ι_{C_2}).

We now turn to the forward direction. If (C_1, ι_{C_1}) and (C_2, ι_{C_2}) are locally equivalent, then by [**Zem17**, Proposition 2.6] $(C_1, \iota_{C_1}) \times (C_2^*, \iota_{C_2}^*)$ is locally equivalent to $(\mathbb{F}[U, U^{-1}], \mathrm{id})$, where $\times$ denotes either $\times_1$ or $\times_2$. Then by Proposition 3.1, $(C_1, \iota_{C_1}) \times (C_2^*, \iota_{C_2}^*)$ is filtered chain homotopy equivalent to $(\mathbb{F}[U, U^{-1}], \mathrm{id}) \oplus (A, \iota_A)$.

Consider $(C_1, \iota_{C_1}) \times (C_2^*, \iota_{C_2}^*) \times (C_2, \iota_{C_2})$. By [**Zem17**, Theorem 1.1], the product $\times$ respects splittings and $(\mathbb{F}[U, U^{-1}], \mathrm{id})$ is the identity element with respect to $\times$. Then

$$\begin{aligned}((C_1, \iota_{C_1}) \times (C_2^*, \iota_{C_2}^*)) \times (C_2, \iota_{C_2}) &\simeq \big((\mathbb{F}[U, U^{-1}], \mathrm{id}) \oplus (A, \iota_A)\big) \times (C_2, \iota_{C_2}) \\ &\simeq (C_2, \iota_{C_2}) \oplus (A', \iota_A'),\end{aligned}$$

where $(A', \iota_A') = (A, \iota_A) \times (C_2, \iota_{C_2})$. Similarly, for some acyclic complex D, we have

$$\begin{aligned}(C_1, \iota_{C_1}) \times ((C_2^*, \iota_{C_2}^*) \times (C_2, \iota_{C_2})) &\simeq (C_1, \iota_{C_1}) \times \big((\mathbb{F}[U, U^{-1}], \mathrm{id}) \oplus (D, \iota_D)\big) \\ &\simeq (C_1, \iota_{C_1}) \oplus (D', \iota_D'),\end{aligned}$$

where $(D', \iota_D') = (D, \iota_D) \times (C_2, \iota_{C_2})$. This concludes the proof of the corollary. □

Using the language of local equivalence, we reprove [**Hom17**, Theorem 1].

Lemma 3.3. *If (C, ι_C) is locally equivalent to $(\mathbb{F}[U, U^{-1}], \mathrm{id})$, then C is filtered chain homotopic to $\mathbb{F}[U, U^{-1}] \oplus A$ for some acyclic complex A.*

Proof of Lemma 3.3. Since (C, ι_C) and $(\mathbb{F}[U, U^{-1}], \mathrm{id})$ are locally equivalent, there exist grading-preserving, filtered chain maps

$$F\colon \mathbb{F}[U, U^{-1}] \to C$$
$$G\colon C \to \mathbb{F}[U, U^{-1}]$$

that induce isomorphisms on homology. Since $\mathbb{F}[U, U^{-1}]$ is isomorphic to its homology, G is surjective and $G \circ F = \mathrm{id}$. Then a standard algebra argument shows that C is filtered isomorphic to $\mathbb{F}[U, U^{-1}] \oplus \ker G$. Namely, $\Phi\colon \mathbb{F}[U, U^{-1}] \oplus \ker G \to C$ given by $(x, y) \mapsto x + y$ and $\Psi\colon C \to \mathbb{F}[U, U^{-1}] \oplus \ker G$ given by $z \mapsto (F \circ G(z), z + F \circ G(z))$ provide the necessary isomorphisms, where we identify $\mathbb{F}[U, U^{-1}]$ with $\operatorname{im} F$. □

Notice that in general ι_C does not respect the splitting in the above lemma. However, we will show that ι_C is homotopic to a map that does split.

Proof of Proposition 3.1. By Lemma 3.3, we may assume C is of the form $\mathbb{F}[U, U^{-1}] \oplus A$. Since (C, ι_C) and $(\mathbb{F}[U, U^{-1}], \mathrm{id})$ are locally equivalent, there exist grading-preserving, filtered chain maps

$$F\colon (\mathbb{F}[U, U^{-1}], \mathrm{id}) \to (C, \iota_C)$$
$$G\colon (C, \iota_C) \to (\mathbb{F}[U, U^{-1}], \mathrm{id})$$

such that $F \circ \mathrm{id} \simeq \iota_C \circ F$ via a skew-filtered chain homotopy H_F and $G \circ \iota_C \simeq \mathrm{id} \circ G$ via a skew-filtered chain homotopy H_G.

We consider the splitting given in Lemma 3.3. Let $p_i\colon \mathbb{F}[U, U^{-1}] \oplus A \to \mathbb{F}[U, U^{-1}] \oplus A$ denote projection onto the i^{th} factor. We have that $p_1 = F \circ G$ and $p_2 = \mathrm{id} + F \circ G$.

Define

$$\iota'_C(x, y) = (x, 0) + p_2 \circ \iota_C(0, y).$$

We claim that $\iota_C \simeq \iota'_C$ via the homotopy $J = H_F \circ G + F \circ H_G \circ p_2$. Indeed,

$$\begin{aligned}
&\iota_C(x, y) + \iota'_C(x, y) \\
&\quad = \iota_C(x, 0) + \iota_C(0, y) + (x, 0) + p_2 \circ \iota_C(0, y) \\
&\quad = \iota_C(x, 0) + \iota_C(0, y) + (x, 0) + \iota_C(0, y) + F \circ G \circ \iota_C(0, y) \\
&\quad = \iota_C \circ F \circ G(x, y) + F \circ \mathrm{id} \circ G(x, y) + F \circ G \circ \iota_C(0, y) + F \circ \mathrm{id} \circ G(0, y) \\
&\quad = \partial \circ H_F \circ G(x, y) + H_F \circ \partial \circ G(x, y) + F \circ \partial \circ H_G(0, y) + F \circ H_G \circ \partial(0, y) \\
&\quad = \partial \circ J(x, y) + J \circ \partial(x, y).
\end{aligned}$$

It is straightforward to check that ι'_C respects the splitting $\mathbb{F}[U, U^{-1}] \oplus A$ and that it is the identity on the first factor, as desired. □

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