Lecture 3 -- Local equivalence, homology cobordism, and applications of involutive

Thursday, July 23, 2026 2:14 PM

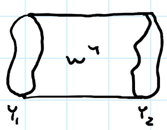
1. $\Theta^3_{\mathbb{Z}}$ and its history
2. Local equivalence
3. $\mathbb{Z}^\infty$-summands
4. Non-generation by SFS's
5. The same ideas elsewhere

Recall An integer homology sphere is a closed oriented 3-mfd Y^3 st $H_*(Y^3;\mathbb{Z}) \cong H_*(S^3;\mathbb{Z})$.

Examples (1) For $K \subseteq S^3$, $S^3_{\pm\frac{1}{n}}(K)$ (by Mayer-Vietoris)

(2) For $(p,q,r)=1$, $\Sigma(p,q,r) = S^5_\varepsilon \cap \{z_1^p + z_2^q + z_3^r = 0\} \subseteq \mathbb{C}^3$

Defn Two integer homology spheres are integer homology cobordant, $Y_1 \sim Y_2$, if there is a smooth cpt W w/ $\partial W = -Y_1 \amalg Y_2$ and $H_*(Y_i;\mathbb{Z}) \xrightarrow{\sim} H_*(W;\mathbb{Z})$ for $i=1,2$.



Examples (1) $K \subseteq S^3$ slice (bounds a smooth disk in B^4), then $S^3_{\pm 1/n}(K) \sim S^3$.

(2) $\Sigma(2,5,7) \sim S^3$. (Mazur, Akbulut)

(3) $Y \# -Y \sim S^3$ by removing a solid ball from $Y \times I$.

The integer homology cobordism group is $\Theta^3_{\mathbb{Z}} \cong \left(\{Y^3 \text{ oriented } \mathbb{Z}HS^3\}, \#\right)/\sim$

- $[Y_1] + [Y_2] = [Y_1 \# Y_2]$ associative, commutative
- $-[Y] = [-Y]$
- $[S^3] = \mathrm{Id}$

Remarks • Smooth necessary. Freedman: Any $\mathbb{Z}HS^3$ bounds an acyclic topological 4-ball.
• $n=3$ necessary • $n=1,2$ $\Theta^n_{\mathbb{Z}}$ trivial
• $n \geq 4$ $\Theta^{PL}_{\mathbb{Z}}$ trivial; $\Theta^n_{\mathbb{Z}} \cong$ homotopy spheres, finite by Kervaire-Milnor

What do we know about this group?

Until the 1980's (Milnor-Rokhlin homomorphism) $\mu: \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}/2\mathbb{Z}$, $[\Sigma(2,3,5)] \longmapsto 1$

$\Sigma(2,3,5) = -S^3_{+1}(RHT)$ Poincaré homology sphere

1989-'91 (Fintushel-Stern, Furuta) $\exists\ \mathbb{Z}^\infty \hookrightarrow \Theta^3_{\mathbb{Z}}$ subgroup

- $\Sigma(2,3,5)$ infinite order; uses Yang-Mills theory

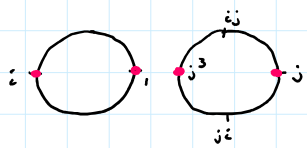
2010 (Furuta) $\exists\ \Theta^3_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}$ summand. Uses analog of d invariant (h) from instanton homology.

'81 (Livingston) Every $\mathbb{Z}HS^3$ is $\sim$ to an irreducible $\mathbb{Z}HS^3$
'83 (Myers) Every $\mathbb{Z}HS^3$ is $\sim$ to a hyperbolic homology sphere.

· 2013 (Manolescu) $\nexists$ an element $[Y]$ of order two having Rokhlin inv't 1.

- Why important? 1970's (Galewski-Stern, Matumoto) This shows $\exists$ topological mfds of every dimn ≥ 5 which do not admit a triangulation (homeomorphism to a simplicial cpx
  - 1, 2, 3 mfds triangulable (2: Radó '25, 3: Moise '52), smooth mfds triangulable (Cairns '35, Whitehead '40)
  - Casson showed Freedman's E_8 mfd nontriangulable ('90)

- What did he use? Pin(2)-equivariant Seiberg-Witten Floer homology



$ij = -ji$ $\rightsquigarrow$ gives rise to three new maps
$j^2 = -1$ $\qquad \alpha, \beta, \gamma : \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ one of which β is integer lift of Rokhlin.

Origin of involutive; under S^1-equivariant SWFH $\cong$ HF (Kutluhan-Lee-Taubes; Colin-Ghiggini-Honda + Taubes, Lidman-Manolescu) the order four action j becomes order two; corresponds to ι_Y.

- HFI is not as powerful as Pin(2)-equivariant SWFH. But its computability has led it add to the study of $\Theta^3_{\mathbb{Z}}$ above.

Back to HFI

- Homology cobordism

does not induce strong equivalence

$F_W : (CF^-(Y_1), \iota_{Y_1}) \longrightarrow (CF^-(Y_2), \iota_{Y_2})$. It induces something we call local equivalence.

Defn An iota-complex is a pair (C, ι) such that

- C is a free, finitely-generated, graded chain cpx over $\mathbb{F}[U]$.
- ι has $\partial\iota + \iota\partial = 0$ (chain map)
  $\iota^2 + Id + \partial H + H\partial = 0$ (squares to identity up to homotopy)

A <u>local map</u> $F : (C_1, \iota_1) \longrightarrow (C_2, \iota_2)$ is a graded chain map st

- $F_* : U^{-1} H_*(C_1) \xrightarrow{\sim} U^{-1} H_*(C_2)$
- $F\iota_1 + \iota_2 F + \partial H + H\partial = 0$ (commutes w/ iota up to htpy)

Two iota-complexes (C_1, ι_1) and (C_2, ι_2) are <u>locally equivalent</u> if they admit local maps in both directions.

$$C_1 \underset{G}{\overset{F}{\rightleftarrows}} C_2$$

<u>Examples</u>

$CF^-(S^3)$ and $CF^-(\Sigma(2,3,7))$

$$\begin{tikzcd} x \arrow[d] & & & a \arrow[d] & & b \arrow[d] \\ Ux \arrow[d] & \arrow[r, bend left, "F ? \times"] & \arrow[l, bend left, "G ? \checkmark"] & Ua \arrow[d] & c \arrow[l] \arrow[r] \arrow[d] & Ub \arrow[d] \\ U^2x & & & U^2a & Uc \arrow[l] \arrow[r] & U^2b \\ & & & & \vdots & \end{tikzcd}$$

$G\rfloor$ $a, b \longmapsto x$ $\quad c \longmapsto 0$

$G\iota + \iota G(a) = G(b) + \iota(x)$
$= x + x$
$= 0$

etc

$F\rfloor$ $x \overset{?}{\longmapsto} a$] has to be a generator

$F\iota(x) + \iota F(x) = F(x) + \iota(a)$
$= a + b$

No possible H exists

$\rightsquigarrow$ Not locally equivalent. Note that local equivalence induces a partial order in which $(CF^-(\Sigma(2,3,7)), \iota_Y) < (CF^-(S^3), \iota_Y)$.

Local equivalence allows us to work w/ conveniently small representatives of the l.e. class.

<u>Exercise</u> $(CF^-(\Sigma(2,3,7) \# \Sigma(2,3,7)), \iota)$ is minimally generated over $\mathbb{F}[U]$ by a model w/ nine generators. Prove that its local equivalence class has a representative w/ five generators over $\mathbb{F}[U]$.

<u>Thm</u> (H.-Manolescu-Zemke) Homology cobordism induces local equivalence.

In particular, there is a homomorphism

$$\Theta^3_{\mathbb{Z}} \longrightarrow \mathfrak{I} := \left(\{(C, \iota) \text{ iota-complexes}\}, \otimes\right)/_{\text{l.e.}}$$

<u>Idea</u> Study structure of $\mathfrak{I}$ to understand $\Theta^3_{\mathbb{Z}}$.
But $\mathfrak{I}$ is pretty bad! For example it has two-torsion elements

③ <u>Major Application I</u> A $\mathbb{Z}^\infty$-summand in $\Theta^3_{\mathbb{Z}}$ (Dai-Hom-Stoffregen-Truong)

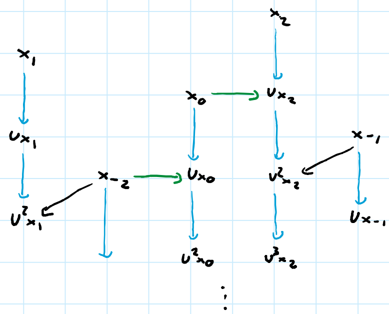
<u>Hope</u> Parametrize l.e. classes. Turns to be useful to draw <u>$\omega = 1 + \iota$</u> instead of ι

<u>Example</u> $CF^-(\Sigma(2,3,7))$ becomes

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & x_1 \arrow[d] & \\ Ux_0 \arrow[d] & Ux_1 \arrow[d] & x_2 \arrow[l] \arrow[d] \\ {} & {} & {} \end{tikzcd}$$

$$\begin{cases} x_0 = a \\ x_1 = a + b \\ x_2 = c \end{cases}$$

Example X



Exercise X is 2-torsion and not l.e. to $CF^-(S^3)$.

New key Idea Weaken local equivalence to "almost" local equivalence

Almost means mod U

Defn An almost iota-complex is a pair $(C, \bar{\iota})$ such that

- C is a free, finitely-generated, graded chain cpx over $\mathbb{F}[U]$.
- $\bar{\iota}$ has $\partial\bar{\iota} + \bar{\iota}\partial = \cancel{0} \in \mathrm{Im}(U)$ (chain map)

  $\bar{\iota}^2 + \mathrm{Id} + \partial H + H\partial = \cancel{0} \in \mathrm{Im}(U)$ (squares to identity up to homotopy)

A local map $F : (C_1, \bar{\iota}_1) \longrightarrow (C_2, \bar{\iota}_2)$ is a graded chain map st

- $F_* : U^{-1}H_*(C_1) \xrightarrow{\sim} U^{-1}H_*(C_2)$
- $F\bar{\iota}_1 + \bar{\iota}_2 F + \partial H + H\partial = \cancel{0} \in \mathrm{Im}(U)$ (commutes w/ iota up to htpy)

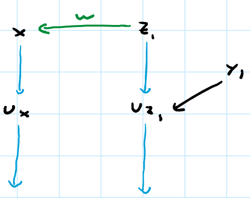
Two iota-complexes $(C_1, \bar{\iota}_1)$ and $(C_2, \bar{\iota}_2)$ are locally equivalent if they admit local maps in both directions.

$$C_1 \underset{G}{\overset{F}{\rightleftarrows}} C_2$$

Ergo $$\Theta^3_{\mathbb{Z}} \longrightarrow \underset{\text{l.e.}}{\mathfrak{I}} \longrightarrow \underset{\text{a.l.e.}}{\widehat{\mathfrak{I}}} = \left(\{(C, \bar{\iota})\}, \otimes\right) / \text{a.l.e.}$$

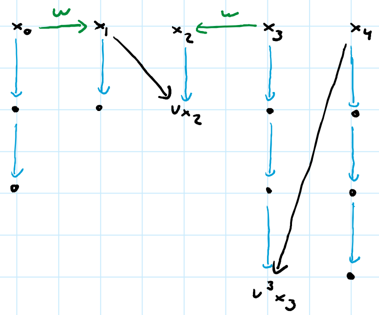
Example For the complex X above, $w \equiv 0 \mod U$. There is a local map from $\mathbb{F}[U]\langle y\rangle$ by sending $y \longmapsto x_0$, and the other direction via $x_0 \longmapsto y$.

Exercise This almost iota complex is not a.l.e. to any iota-cpx.



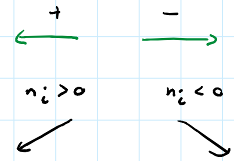
$C(+,1)$

Defn A standard complex is the almost iota-complex generated by $x_0, x_1, \dots, x_{2n}$ and a sequence $(\pm, n_1, \pm, n_2, \pm, n_3, \dots)$



$C(-, -1, +, +3)$

$\Sigma(2,3,7) \rightsquigarrow C(-,-1)$



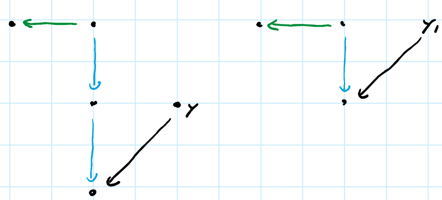
Thm (DHST) Every almost iota-cpx is a.l.e. to a std cpx. Moreover,

$\varphi_i = \{\#\text{ parameters equal to } i\} - \{\#\text{ parameters equal to } -i\}$ is a homomorphism $\mathfrak{I} \longrightarrow \mathbb{Z}$

Thm (DHST) $\mathfrak{I}$ is totally ordered by ordering the std complexes lexicographically wrt the order $\begin{cases} - <' + \\ -1 <' -2 <' -3 <' \dots <' 0 <' \dots <' 3 <' 2 <' 1 \end{cases}$

Why this funny ordering?

e.g. $C(+, 2) \longrightarrow C(+, 1)$



via $y \longmapsto Uy_1$

One can check this gives linearly independent maps $\varphi_i : \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}$, verifying that we get $\Theta^3_{\mathbb{Z}} \to \mathbb{Z}^\infty$ a summand.

④ **Major Application II** Non-generation of $\Theta_{\mathbb{Z}}$ by SFS's (H.-Hom-Stoffregen-Zemke

Previously (Stoffregen, Lin, Frøyshov) $\exists$ a $\mathbb{Z}HS^3$ Y st Y is not homology cobordant to any Seifert fibred sphere.

Pin(2)-SWFH ↑ Pin(2)-monopole ↑ Mod 2 instanton homology ↑

Stoffregen's example $\Sigma(2,3,11) \# \Sigma(2,3,11)$

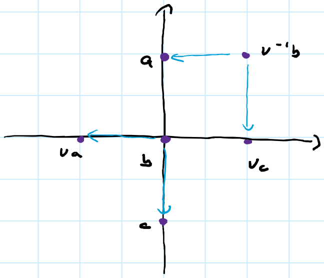
(Nozaki-Sato-Taniguchi) One can choose an example which is not $\sim S^3_{+1/n}(K)$ either.

Q Is the subgroup Θ^3_{SFS} generated by the Seifert fibred spheres equal to $\Theta^3_{\mathbb{Z}}$?

(DHST) Computed image
$$\begin{array}{ccc} \Theta^3_{\mathbb{Z}} & \longrightarrow & \mathfrak{I} \\ \cup\!\!| & \nearrow & \\ \Theta^3_{SFS} & & \end{array}$$
. Every element in the image has standard cpx representative $C(\mp, n_1, \mp, n_2, \mp, n_3, \dots)$ such that $n_1 \geq n_2 \geq n_3 \geq \dots$

So it suffices to realize any std cpx not of this form by a $\mathbb{Z}HS^3$, or better yet a family $C(+, -1, +, -n)$

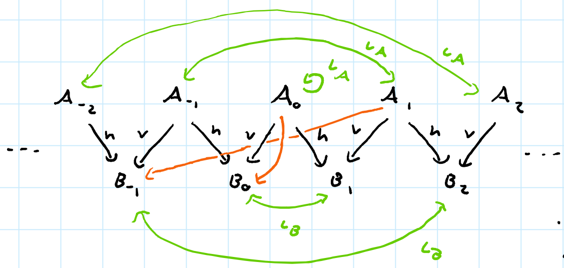
New key Tool: The Full surgery Formula.



Thm (HHSZ) $(CF^-(S^3_{p/q}(K)),\iota)$ is determined by $(CFK^\infty(K),\iota_K)$

$\iota_A = U^{-s}\iota_K : A_s \longrightarrow A_{-s}$ $\qquad\qquad$ $\iota_B = \iota_K F : B \circlearrowleft$

Example (1-surgery)



$\iota_\# = \iota_A + H + \iota_B$

$$\begin{tikzcd}
A^- \arrow[r, "D_n"] \arrow[d, "\iota_A"'] \arrow[dr, "H"] & B^- \arrow[d, "\iota_B"] \\
A^- \arrow[r, "D_n"'] & B^-
\end{tikzcd}$$

- $\partial H + H\partial = \iota_B h + v\iota_A$
- Can always choose one composition to commute

Simple up to l.e. Thm (HHSZ) $(CF^-(S^3_{+1}(K)),\iota) \underset{l.e.}{\simeq} (A_0^-,\iota_K)$

Remark $\underline{V}_0(K) = \frac{-1}{2}\underline{d}(S^3_{+1}(K))$ and $\bar{V}_0(K) = \frac{-1}{2}\bar{d}(S^3_{+1}(K))$ are knot concordance invts computably directly from A_0^-.

How to build up interesting examples?

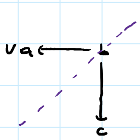
- $\exists$ a map $\left\{\begin{array}{l}\iota_K\text{-complexes}\\ \text{up to } \iota_K\text{-local equivalence}\end{array}\right\} \xrightarrow[+1\text{ surgery}]{} \left\{\begin{array}{l}\text{(almost) } \iota\text{-complexes}\\ \text{up to (almost) local equivalence}\end{array}\right\}$

Tensor product formula here is complicated even for simple inputs.

Thm (Zemke) $(CFK^\infty(K_1 \# K_2), \iota_{K_1\#K_2}) \simeq (CFK^\infty(K_1) \otimes CFK^\infty(K_2), (Id \otimes Id + \Psi \otimes \Phi)\iota_{K_1} \otimes \iota_{K_2})$

vertically odd length arrows $\qquad$ Horizontally odd length arrows

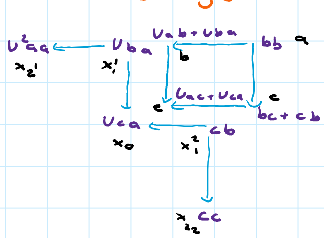
Example



$\Psi \otimes \Phi(bb) = Uca$, else zero

$\iota_{RHT\#RHT}$:
$bb \longmapsto bb + Uca$
$Uba \longleftrightarrow bc$ $\qquad$ $cc \longleftrightarrow U^2aa$
$Uab \longleftrightarrow Uac$ $\qquad$ $Vac \longleftrightarrow Uca$

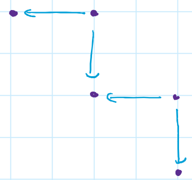
Exercise With a change of basis, this becomes



$a \longmapsto a + x_0$ $\qquad$ $b \longleftrightarrow c$
$x_0 \longmapsto x_0 + e$ $\qquad$ $x_1' \longleftrightarrow x_1^2$
$e \circlearrowleft$ $\qquad$ $x_2' \longleftrightarrow x_2^2$

Exercise $(CFK^\infty((RHT \# RHT) \# \overline{T(2,5)}), \iota_K) \simeq (CFK^\infty(4_1), \iota_K)$

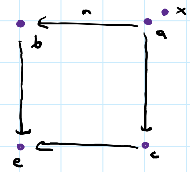
T(2,5)



That is, up to local equivalence tensoring w/ dual of staircase removes it leaving the box.

We show $\#_2 T(2n,2n+1) \# \overline{T(4n,2n+1)}$ for n odd is an $n \times n$ box cpx w/ same maps as 4_1.

Exercise If B_n =



n odd

then $CF^-(S^3_{+1}(B_n^\vee \otimes T(2,3)), \iota) \underset{a.e.e.}{\simeq} C(+1,-1,+1,-n)$.

Thm (HHSZ) The classes $[S^3_{+1}(\#_2 \overline{T(2n,2n+1)} \# T(2n,4n+1) \# T(2,3))]$ generate a subgroup $\mathbb{Z}^\infty \subseteq \Theta^3_{\mathbb{Z}}/\Theta_{SFS}$.

(5) The same ideas elsewhere

Knot Concordance

- [DHST] Notion of non-involutive almost l.e. used to give a new proof that concordance group $\mathcal{C}$ has a $\mathbb{Z}^\infty$-summand.
- [Zhou] Same techniques used to give $\mathbb{Z}^\infty$-summand of "homology concordance" group ($\mathbb{Z}HS^3$ + knot)

Actions on Knots & 3-mfds

- [Dai-Hedden-Mallick] studied $\Theta^{3,\tau}_{\mathbb{Z}}$ equivariant homology cobordism group using ι and τ induced by action; gave many new strong corks.
- [Alfieri-Kang-Stipsicz] studied $\mathcal{C}$ via $\mathcal{C} \xrightarrow[\Sigma(K)]{} \Theta^3_{\mathbb{Z}/2}$ and used $(CF^-(\Sigma(K)), \tau)$
- [Dai-Mallick-Stoffregen] studied l.e. of actions on symmetric knots
  - [Mallick] large surgery formula $(K, \tau_K) \rightsquigarrow (S^3_n(K), \tau)$
  - [Dai-Kang-Mallick-Park-Stoffregen] Used interplay of actions to show (2,1)-cable of 4_1 not smoothly slice.
    ↳ since extended using real SWFH.
  - [H.-Mallick-Stoffregen-Zemke] Full surgery formula for τ; $\mathbb{Z}^\infty$ summand in $\Theta^{3,\tau}_{\mathbb{Z}}$.

Khovanov theory

- [Dunfield-Lipshitz-Schuetz, Borodzik-Dai-Mallick-Stoffregen] Local equivalence theories for Kh of symmetric knots.
- [Sano] Mapping cone theory for Kh of symmetric knots.