Lecture 2 -- Involutive HF

Thursday, July 23, 2026 2:14 PM

0. Naturality (recall)
1. construction & first examples
2. The large surgery formula
3. Cobordisms & notions of equivalence

0. ~~Recall~~ <u>Naturality</u> (Y, w, s) a 3-mfd. Say we have two sets of Heegaard data $\mathcal{H}_1, \mathcal{H}_2$

$$\begin{tikzcd}
 & \mathcal{H}_i \arrow[dr] & \\
\mathcal{H}_1 \arrow[ur] \arrow[dr] & & \mathcal{H}_2 \\
 & \mathcal{H}_j \arrow[ur] &
\end{tikzcd}$$

We can imagine going from $\mathcal{H}_1$ to $\mathcal{H}_2$ along multiple paths of Heegaard moves.

<u>Thm</u> (Juhász - Thurston - Zemke) There are canonical chain homotopy equivelences $\Phi(\mathcal{H}_1, \mathcal{H}_2) : CF^\circ(\mathcal{H}_1) \longrightarrow CF^\circ(\mathcal{H}_2)$ induced by the Reidemeister - Singer moves st

1. $\Phi(\mathcal{H}_2, \mathcal{H}_3) \circ \Phi(\mathcal{H}_1, \mathcal{H}_2) \simeq \Phi(\mathcal{H}_1, \mathcal{H}_3)$
2. $\Phi(\mathcal{H}, \mathcal{H})$.

One can then take a direct limit of this transitive system to show $CF^\circ(Y, z, s)$ is a well-defined chain homotopy equivalence class of chain cpxes and $HF^\circ(Y, z, s)$ is a well-defined module.

1. <u>Involutive Heegaard Floer homology</u> (H-Manolescu, H-Manolescu Zemke)

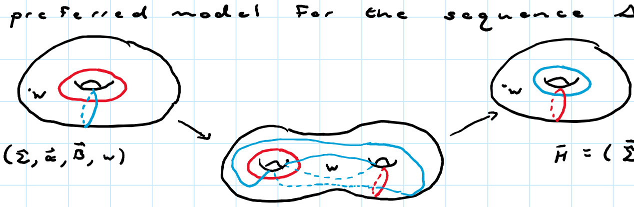
Recall there is a conjugation map on $Spin^c(Y)$, $s \longmapsto \bar{s}$. $\exists$ a lift of this action defined as follows.

$\mathcal{H} = (H, J_t)$ $\quad$ $\bar{\mathcal{H}} = (\bar{H}, -J_{1-t})$ $\quad$ $\mathcal{H} = (H, J_t)$

$H = (\Sigma, \vec{\alpha}, \vec{\beta}, w)$ $\quad$ $\bar{H} = (\bar{\Sigma}, \vec{\beta}, \vec{\alpha}, w)$

$$\iota_{\mathcal{H}} : CF^\circ(\mathcal{H}, s) \xrightarrow{\ \eta\ } CF^\circ(\bar{\mathcal{H}}, \bar{s}) \xrightarrow{\ \Phi(\bar{\mathcal{H}}, \mathcal{H})\ } CF^\circ(\mathcal{H}, \bar{s})$$

<u>Notes</u>
1. First step is replacing F w/ 3-F.
2. $\exists$ a preferred model for the sequence Δ of Heegaard moves:



$H = (\Sigma, \vec{\alpha}, \vec{\beta}, w)$ $\quad$ $\bar{H} = (\bar{\Sigma}, \vec{\beta}, \vec{\alpha}, w)$

$D(H) = (\Sigma \# \bar{\Sigma}, \vec{\alpha} \cup \vec{\beta}, \Delta, w)$

Lemma 1 $\iota_{\mathcal{H}}^2 \simeq \mathrm{Id}.$

Proof $\iota_{\mathcal{H}^2} \simeq \Phi(\bar{\mathcal{H}},\mathcal{H})\circ \underline{\eta \circ \Phi(\bar{\mathcal{H}},\mathcal{H})\circ \eta}$. As $\eta^{-1}=\eta$, the term in red is conjugation, which gives the map associated to the conjugate set of Heegaard moves from $\mathcal{H}$ to $\bar{\mathcal{H}}$.

$$= \Phi(\bar{\mathcal{H}},\mathcal{H})\circ \Phi(\mathcal{H},\bar{\mathcal{H}})$$

$$\simeq \mathrm{Id}. \quad \square$$

We focus on $\mathfrak{s}=\bar{\mathfrak{s}}$ self-conjugate.

Invariance We want to know that given $\mathcal{H}$ and $\mathcal{H}'$ for (Y,z), the preferred map $\Phi(\mathcal{H},\mathcal{H}') : CF^\circ(\mathcal{H}) \longrightarrow CF^\circ(\mathcal{H}')$ is equivariant up to htpy, i.e.

$$\Phi(\mathcal{H},\mathcal{H}')\circ \iota_{\mathcal{H}} \simeq \iota_{\mathcal{H}'} \circ \Phi(\mathcal{H},\mathcal{H}').$$

Sometimes called "strong equivalence".

$$\Phi(\mathcal{H},\mathcal{H}')\circ\Phi(\bar{\mathcal{H}},\mathcal{H})\circ\eta \qquad\qquad \Phi(\bar{\mathcal{H}}',\mathcal{H}')\circ\eta\circ\Phi(\mathcal{H},\mathcal{H}')$$

$$\Phi(\bar{\mathcal{H}},\mathcal{H}')\circ\eta \qquad\qquad \Phi(\bar{\mathcal{H}}',\mathcal{H}')\circ\Phi(\bar{\mathcal{H}},\bar{\mathcal{H}}')\circ\eta$$

$$\Phi(\bar{\mathcal{H}},\mathcal{H}')\circ\eta$$

A more sophisticated packaging:

```tikzcd
CF^-(\mathcal{H},\mathfrak{s}) \arrow[r, "\Phi(\mathcal{H},\mathcal{H}')"] \arrow[d, "\eta"] \arrow[ddr, dashed, "G"] \arrow[dr, phantom, "\circlearrowleft"] & CF^-(\mathcal{H}',\mathfrak{s}) \arrow[d, "\eta"] \\
CF^-(\bar{\mathcal{H}},\bar{\mathfrak{s}}) \arrow[r, "\Phi(\bar{\mathcal{H}},\bar{\mathcal{H}}')"] \arrow[d, "\Phi(\bar{\mathcal{H}},\mathcal{H})"] \arrow[dr, dashed, "H"] & CF^-(\bar{\mathcal{H}}',\bar{\mathfrak{s}}) \arrow[d, "\Phi(\bar{\mathcal{H}}',\mathcal{H}')"] \\
CF^-(\mathcal{H},\mathfrak{s}) \arrow[r, "\Phi(\mathcal{H},\mathcal{H}')"'] & CF^-(\mathcal{H}',\bar{\mathfrak{s}})
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In the general case H is not known to be independent of the choices defining it; in this specific case it can be pinned down after framing T_zY. (H-Hom-Stoffregen-Zemke)

From now on $(CF^-(Y,\mathfrak{s}), \iota_Y)$.

Examples

$CF^-(S^3)$

```tikzcd
x \arrow[d] \arrow[loop right, "\iota"] \\
Ux \arrow[d] \arrow[loop right, "\iota"] \\
U^2x \arrow[loop right, "\iota"] \\
\vdots
```

$HF^-(S^3) \simeq \mathbb{F}_{(0)}[U]$

```tikzcd
{[x]} \arrow[d] \arrow[loop right, "\iota_*"] \\
U[x] \arrow[d] \arrow[loop right, "\iota_*"] \\
U^2[x] \arrow[loop right, "\iota_*"] \\
\vdots
```

$CF^-(\Sigma(2,3,7))$

```tikzcd
a \arrow[rr, leftrightarrow, "\iota"] \arrow[d] & & b \arrow[d] \\
& c \arrow[loop right] \arrow[dl] \arrow[dr] \arrow[d, dashed] & \\
Ua \arrow[rr, leftrightarrow] \arrow[d] & Uc \arrow[loop right] \arrow[dl] \arrow[dr] & Ub \arrow[d] \\
U^2a \arrow[rr, leftrightarrow] & & U^2b \\
& \vdots &
```

$HF^-(\Sigma(2,3,7)) \simeq \mathbb{F}_{(0)}[U] \oplus \mathbb{F}_{(0)}$

```tikzcd
{[a]} \arrow[rr, leftrightarrow, "\iota_*"] \arrow[dr] & & {[b]} \arrow[dl] \\
& U[a] \arrow[d] \arrow[loop right, "\iota_*"] & \\
& U^2[a] \arrow[loop right] & \\
& \vdots &
```

In sufficiently low gradings $\iota_* = \mathrm{Id}$.

Two ways to Package This Data ① Form a mapping cone; ② Study the pair $(CF^-(Y,s), c_Y)$.

① Form a mapping cone $CFI^-(Y,s) = \mathrm{Cone}\left(CF^-(Y,s) \xrightarrow{Q(1+c_Y)} Q\,CF^-(Y,s)[-1]\right)$

So as a vector space the complex is $CF^-(Y,s)[-1] \oplus Q\,CF^-(Y,s)[-1]$, with differential $\begin{pmatrix} \partial & 0 \\ Q(1+c_Y) & \partial \end{pmatrix}$.

$\deg Q = -1$; Module over $\mathbb{F}_2[U,Q]/Q^2 = H^*(B\mathbb{Z}_4; \mathbb{F}_2)$

The discussion above becomes the existence of a che

$$CFI^-(\mathcal{H},s) \xrightarrow{\Phi_i} CFI^-(\mathcal{H}',s)$$

$$(a, Qb) \longmapsto (\Phi(\mathcal{H},\mathcal{H}')(a),\ Q(G(a)) + \Phi(\mathcal{H},\mathcal{H}')(b))$$

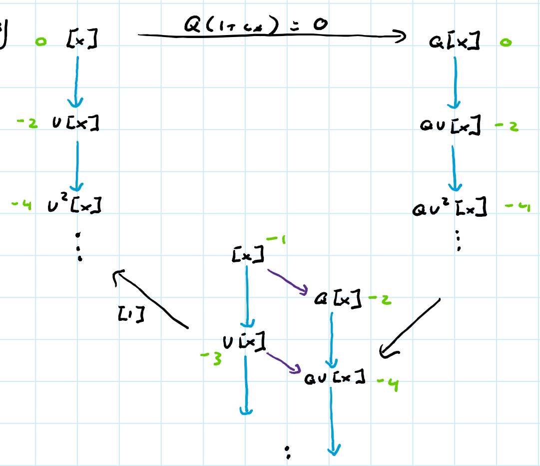
$$\Phi_i = \begin{pmatrix} \Phi(\mathcal{H},\mathcal{H}') & 0 \\ QG & \Phi(\mathcal{H},\mathcal{H}') \end{pmatrix}$$

Exercise This is a chain map.

This naturally gives us an exact triangle.

$$\begin{tikzcd}
HF^-(Y,s) \arrow[rr, "Q(1+c_Y)"] & & QHF^-(Y,s)[-1] \arrow[dl] \\
& HFI^-(Y,s) \arrow[ul] &
\end{tikzcd}$$

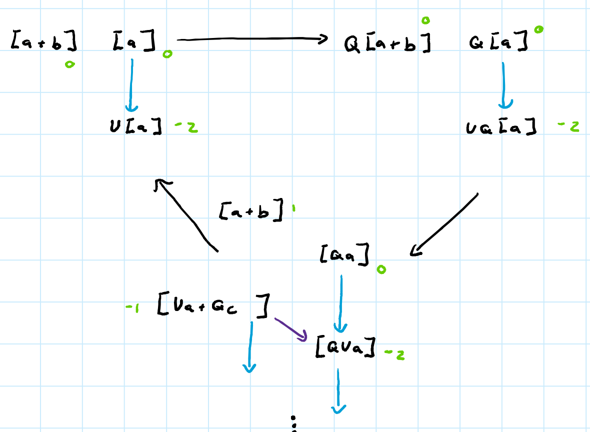
Examples S^3



· Observe that there are necessarily two towers in $HFI^-(Y,s)$.

· Homological gradings in lime.

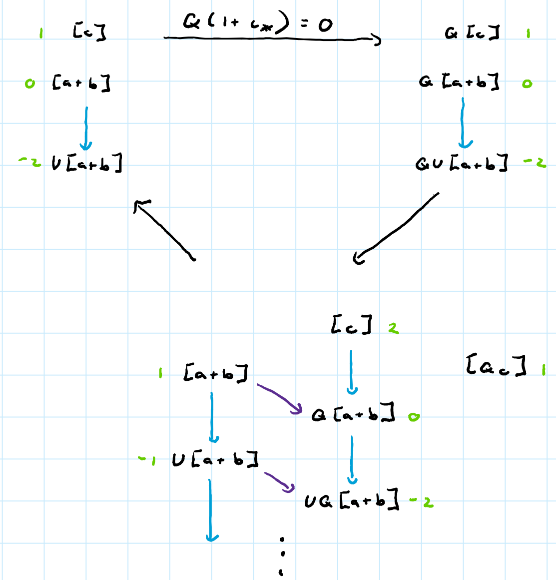
$\Sigma(2,3,7)$



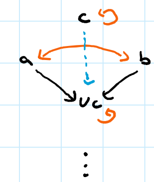
· Lefthand tower can be shortened by cancellation.

$$\begin{tikzcd}
& Qc \arrow[d] \\
Ua \arrow[r] & QU(a+b)
\end{tikzcd}$$

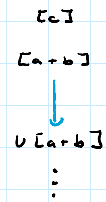
$-\Sigma(2,3,7)$

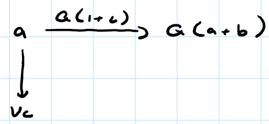


$CF^-(-\Sigma(2,3,7))$



$HF^-(-\Sigma(2,3,7)) \simeq \mathbb{F}_{(0)}[U] \oplus \mathbb{F}_{(1)}$





$\rightsquigarrow [Uc] = Q[a+b]$

Defn Given a QHS^3 Y and $s \in Spin^c(Y)$ st $s = \bar{s}$, we have

$$\underline{d}(Y,s) = \max\{r : HFI_r^-(Y) \text{ contains } [x] \neq 0 \text{ st } U^n[x] \neq 0 \text{ for all } n \geq 0 \text{ and } U^n[x] \notin Im(Q) \text{ for any } n\} - 1$$

$= $ Top grading of lefthand tower -1

$$\bar{d}(Y,s) = \max\{r : HFI_r^-(Y) \text{ contains } [x] \neq 0 \text{ st } U^n[x] \neq 0 \text{ for all } n \geq 0 \text{ and } U^n[x] \in Im(Q) \text{ for some } n \geq 0\}$$

$=$ Top grading of righthand tower

Example • $\underline{d}(S^3) = d(S^3) = \bar{d}(S^3) = 0$ Likewise all three agree for any Heegaard Floer L-space.

• $\underline{d}(\Sigma(2,3,7)) = -2$; $d(\Sigma(2,3,7)) = \bar{d}(\Sigma(2,3,7)) = 0$

• $\underline{d}(-\Sigma(2,3,7)) = d(-\Sigma(2,3,7)) = 0$; $\bar{d}(\Sigma(2,3,7)) = 2$.

Exercise • From U-equivariance of the exact triangle, $\underline{d}(Y,s) \leq d(Y,s) \leq \bar{d}(Y,s)$

• The U-torsion order of $HF_{red}(Y,s)$ bounds $\frac{1}{2}(d(Y,s) - \underline{d}(Y,s))$ and $\frac{1}{2}(\bar{d}(Y,s) - d(Y,s))$

Thm [H.-Manolescu-Zemke] $\simeq$ is "strong equivalence", i.e. equivariant che.

- $(CF^-(-Y,s), \iota_Y) \simeq (\mathrm{Hom}_{\mathbb{F}[U]}(CF^-(Y,s), \mathbb{F}[U]), \iota_Y^*)$
- $(CF^-(Y_1 \# Y_2), \iota_{Y_1 \# Y_2}) \simeq (CF^-(Y_1) \otimes_{\mathbb{F}[U]} CF^-(Y_2), \iota_{Y_1} \otimes \iota_{Y_2})$

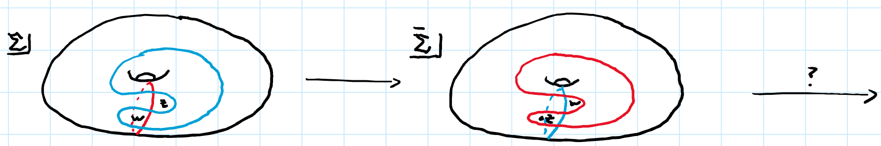
Note that dual over $\mathbb{F}_2$ is CF^+ instead.

Statements w/o ι are Ozsváth-Szabó.

How did I know those examples?

(2) First computational tool: large surgery formula (H.-Manolescu)

Apply ideas of construction of ι to a knot diagram.



$\mathcal{H} = (H, J_s)$ $\qquad$ $\bar{\mathcal{H}} = (\bar{H}, -J_{1-s})$

$H = (\Sigma, \vec{\alpha}, \vec{\beta}, z, w)$ $\qquad$ $\bar{H} = (\bar{\Sigma}, \vec{\beta}, \vec{\alpha}, w, z)$

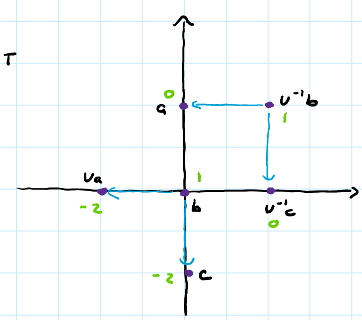
Must do half a Dehn twist about the knot to return basepoints.

$$\iota_K : CFK^\infty(\mathcal{H}) \xrightarrow{\eta_K} CFK^\infty(\bar{\mathcal{H}}) \xrightarrow{\Phi_{\bar{\mathcal{H}}, \rho\mathcal{H}}} CFK^\infty(\rho\mathcal{H}) \xrightarrow{\eta_\rho} CFK^\infty(\mathcal{H})$$

We get a map ι_K st
- ι_K chain map, preserves homological grading, reverses $\mathbb{Z} \oplus \mathbb{Z}$-filtration.
- $\iota_K^2 \simeq \xi_K$ the Sarkar Dehn twist map.
- $\iota_K^4 \simeq \mathrm{Id}$

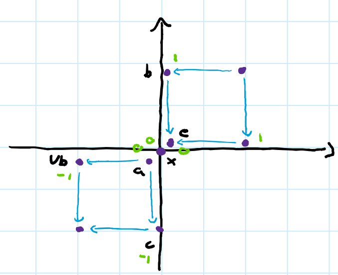
Examples

(1) RHT



- ξ_K: Id
- ι_K: $a \longleftrightarrow U^{-1}c$, $b \circlearrowleft$

(2) 4_1



- ξ_K: $a \longmapsto a+e$, $b \circlearrowleft$ $c \circlearrowleft$ $e \circlearrowleft$ $x \circlearrowleft$

ι_K: $a \longmapsto a+x$, $x \longmapsto x+e$, $c \longleftrightarrow Ub$, $e \circlearrowleft$

Exercises (1) Up to change of basis, ι_K is the unique map w/ $\iota_K^2 \simeq \xi$.

(2) W/o filtration, ξ_K is homotopic to Id, but there is no filtered homotopy.

Observe $A_0^- = \mathcal{F}_{(0,0)}$ is always preserved by ι_K.

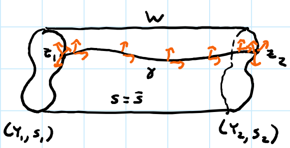
Large surgery Formula (H.-Manolescu) Let $n \geq g(K)$. Recall $\mathrm{Spin}^c(S^3_n(K)) \simeq \mathbb{Z}/n\mathbb{Z}$.
$\overline{[i]} = [-i]$

$$(CF^-(S^3_n(K), [0]), \iota_Y) \underset{\text{equivariant che}}{\simeq} (A_0^-[d_n], \iota_K) \qquad d_n = d(L(n,1), [0])$$
$$= \frac{n-1}{4} \quad \text{Importantly zero for } n=1.$$

Exercise $\Sigma(2,3,7) = S^3_{+1}(4_1)$. From this, extract the computation of $(CF^-(\Sigma(2,3,7)), \iota_Y)$ above.

More computations A SFS always has Heegaard Floer homology of a symmetric "graded root" structure, combinatorially computable. Dai-Manolescu proved $(CF^-(Y), \iota_Y)$ is also combinatorial and Dai-Stoffregen computed connect sums of such classes (w/ either orientation).

Cobordism maps in HFI



Say W is a cobordism from Y_1 to Y_2 with a self-conjugate spin^c structure s w/ $s|_{Y_i} = s_i$. Given a framed path (γ, z) connecting z_1, z_2, $\exists$

$$CFI(W, \gamma, z, s) : CFI^-(Y_1, z_1, s_1) \longrightarrow CFI^-(Y_2, z_2, s_2)$$

given by
$$\begin{tikzcd} CF^-(Y_1, z_1, s_1) \arrow[r, "F(W,\gamma,s)"] \arrow[d, "\iota_{Y_1}"] \arrow[dr, "H"] & CF^-(Y_2, z_2, s_2) \arrow[d, "\iota_{Y_2}"] \\ CF^-(Y_2, z_2, s_2) \arrow[r, "F(W,\gamma,s)"] & CF^-(Y_2, z_2, s_2) \end{tikzcd}$$

These have the same grading changes as in HF.

What kind of maps between pairs (C, ι) should we care about?
Proposed Whatever kind of map is produced by (mostly integer) homology cobordism.