Lecture 1 -- Intro to Heegaard Floer homology

Tuesday, August 18, 2026 1:12 PM

1. What is it?
2. How do we construct it?
3. What is its history?
4. What are its advantages?

① What is it? (Ozsváth - Szabó, 2000s)

$(Y, w, \mathfrak{s}) \rightsquigarrow CF^-(Y, w, \mathfrak{s}) \rightsquigarrow HF^-(Y, w, \mathfrak{s})$

Y a 3-mfd, $\mathfrak{s} \in Spin^c(Y)$, w basepoint

$CF^-(Y,w,\mathfrak{s})$: Free, finitely gen'd chain complex over $\mathbb{F}[U]$

$HF^-(Y,w,\mathfrak{s})$: Module over R

- Also versions $\{\wedge, +, \infty\}$
- $\mathbb{F} = \mathbb{F}_2$, $\deg(U) = -2$
- $\mathbb{F}[U] \simeq H^*(BS^1; \mathbb{F}_2)$, $BS^1 = \mathbb{CP}^\infty$
- We will drop basepoint from notation often.

<u>If Y a $\mathbb{Q}HS^3$</u> (i.e. $H^*(Y;\mathbb{Q}) \simeq H^*(S^3;\mathbb{Q})$) then relatively $\mathbb{Z}$-graded, absolutely $\mathbb{Q}$-graded.

(Ozsváth - Szabó) $HF^-(Y,\mathfrak{s}) \simeq \mathbb{F}_{(d)}[U] \oplus \left(\bigoplus_{i=1}^{m} \mathbb{F}_{(n_i)}[U]/U^{n_i}\right)$

<u>Examples</u>

$CF^-(S^3)$:
$$\begin{tikzcd} x \;\; 0 \arrow[d] \\ Ux \;\; -2 \arrow[d] \\ U^2x \;\; -4 \end{tikzcd}$$

$HF^-(S^3) \simeq \mathbb{F}_{(0)}[U]$:
$$\begin{tikzcd} {[x]} \;\; 0 \arrow[d] \\ U[x] \;\; -2 \arrow[d] \\ U^2[x] \;\; -4 \end{tikzcd}$$

$CF^-(\Sigma(2,3,5))$:
$$\begin{tikzcd} y \;\; 2 \arrow[d] \\ Uy \;\; 0 \arrow[d] \\ U^2y \;\; -2 \end{tikzcd}$$

$HF^-(\Sigma(2,3,5)) \simeq \mathbb{F}_{(2)}[U]$:
$$\begin{tikzcd} {[y]} \;\; 2 \arrow[d] \\ U[y] \;\; 0 \arrow[d] \\ U^2[y] \;\; -2 \end{tikzcd}$$

$CF^-(\Sigma(2,3,7))$:
$$\begin{tikzcd} 0 \;\; a \arrow[d] & & b \;\; 0 \arrow[d] \\ & c \;\; -1 \arrow[dl] \arrow[dr] \arrow[d] & \\ -2 \;\; Ua \arrow[d] & Uc \arrow[dl] \arrow[dr] & Ub \;\; -2 \arrow[d] \\ U^2a & & U^2b \end{tikzcd}$$

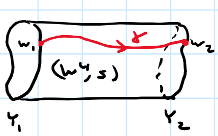
$HF^-(\Sigma(2,3,7)) \simeq \mathbb{F}_{(0)}[U] \oplus \mathbb{F}_{(0)}$:
$$\begin{tikzcd} {[a]} \arrow[dr] & & {[b]} \arrow[dl] \\ & U[a] \arrow[d] & \\ & U^2[a] & \end{tikzcd}$$

<u>Rmk</u>

$\Sigma(p,q,r) \simeq \{z_1^p + z_2^q + z_3^r\} \cap S^5_\varepsilon \subseteq \mathbb{C}^3$

$(p,q,r) = 1 \rightsquigarrow H^*(\Sigma(p,q,r);\mathbb{Z}) \simeq H^*(S^3;\mathbb{Z})$

<u>Cobordism maps</u>



- $F(W^4, \mathfrak{s}, \gamma) : HF^-(Y_1, \mathfrak{s}|_{Y_1}, w_1) \longrightarrow HF^-(Y_2, \mathfrak{s}|_{Y_2}, w_2)$
- Change of grading determined by topology.

<u>Powerful numerical invariants</u>

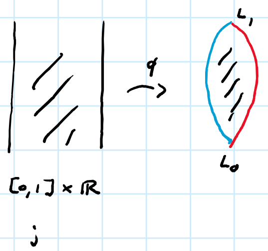
- Y a $\mathbb{Q}HS^3 \rightsquigarrow d(Y,\mathfrak{s}) = gr(1)$
  $d(S^3) = d(\Sigma(2,3,7)) = 0$
  $d(\Sigma(2,3,5)) = 2$
- Four mfd invt from cobordism maps.

② How do we construct it?

Lagn Floer homology (M^{2n}, ω) symplectic ω closed nongenerate 2-form

L_0, L_1 Lagrangians n-dimensional, $\omega|_{L_i} = 0$

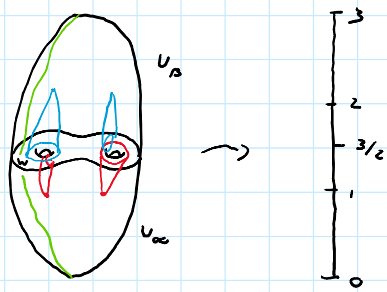
Floer (M, L_0, L_1) + hypotheses $\rightsquigarrow CF(M, L_0, L_1) = (\mathbb{F}_2\langle L_0 \pitchfork L_1 \rangle, \partial)$ $\partial x = \sum_{\mu(\phi)=1} \#\widehat{m}(\phi)\, y$



J_t compatible w/ ω

For a 3-mfd

Choose a self-indexing Morse fcn.



$\Sigma = f^{-1}(3/2)$

$\vec{\alpha} = (\alpha_1, \dots, \alpha_g)$

$\vec{\beta} = (\beta_1, \dots, \beta_g)$

$H = (\Sigma, \vec{\alpha}, \vec{\beta}, w)$ $\mathcal{H} = (H, J_t)$

J_t cptble w/ a choice of ω_0 on Σ

- $M = Sym^g(\Sigma)$
- Complex mfd via $Sym^n(\mathbb{C}) \longleftrightarrow \mathbb{C}^n$

  $(z-a_1)\cdots(z-a_n) \longleftrightarrow z^n + b_{n-1}z^{n-1} + \dots + b_0$
- $\exists\ \omega$ on $Sym^g(\Sigma)$ cptble w/ $Sym^g(J_t)$ st ω agrees w/ $\omega_0^{\times n}$ away from fat diagonal [Perutz]

- $L_0 = \mathbb{T}_{\vec{\alpha}} = \alpha_1 \times \dots \times \alpha_g$
- $L_1 = \mathbb{T}_{\vec{\beta}} = \beta_1 \times \dots \times \beta_g$

- Generators are g-tuples of points st each α, β is used once.

- $CF^-(\mathcal{H}) = (\mathbb{F}[U]\langle \mathbb{T}_\alpha \pitchfork \mathbb{T}_\beta \rangle, \partial)$

$$\partial x = \sum_{\mu(\phi)=1} U^{n_w(\phi)} \#\widehat{m}(\phi)\, y$$

$V_w = \{w\} \times Sym^{g-1}(\Sigma)$

$n_w(\phi) = \#(\mathrm{Im}(\phi) \pitchfork V_w)$

- Set $U=0 \rightsquigarrow \widehat{CF}(\mathcal{H}) = CF(Sym^g(\Sigma \setminus w), \mathbb{T}_\alpha, \mathbb{T}_\beta)$

Warning Requires an admissibility condition. For a $\mathbb{Q}HS^3$, can be satisfied for all spin^c structures at once.

Examples

S^3

$\mathcal{H}_1$

$CF^-(\mathcal{H}_1) = \mathbb{F}[v]\langle x\rangle$, $\partial = 0$

$\mathcal{H}_2$

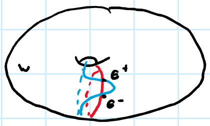
$CF^-(\mathcal{H}_2) = \mathbb{F}[v]\langle a, b, c\rangle$

$\partial b = va + c$

$HF^-(\mathcal{H}_2) \simeq \mathbb{F}_0[v]$ gen'd by $[c]$

$S^1\times S^2, s_0$

$\mathcal{H}_3$

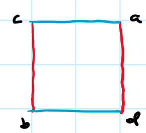


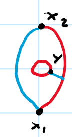
$CF^-(\mathcal{H}_3, s_0) \simeq \mathbb{F}[v]\langle\theta^+, \theta^-\rangle$, $\partial\theta^+ = \theta^- + \theta^- = 0$

$HF^-(\mathcal{H}_3, s_0) \simeq \mathbb{F}_{(1/2)}[v] \oplus \mathbb{F}_{(-1/2)}[v]$

Remark Counting curves on genus 1 is easy, but this rapidly becomes much harder.

$$\begin{tikzcd} \tilde{\Sigma}(D^2) \arrow[r] \arrow[d] & \Sigma \times Sym^{g-1}(\Sigma) \arrow[d] \\ D^2 \arrow[r] & Sym^g(\Sigma) \end{tikzcd}$$

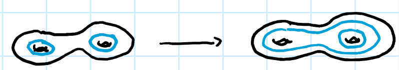


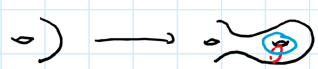


$ab \longmapsto cd$

Invariance $\mathcal{H}_1, \mathcal{H}_2$ for (Y, s) related by

Reidemeister-Singer
(i) Isotopy of curves
(ii) Handleslides
(iii) Stabilizations
(iv) Changes to acs





Ozsváth-Szabó (i)-(iv) give chain homotopy equivalences $\Phi(\mathcal{H}_1, \mathcal{H}_2) : CF^-(\mathcal{H}_1, s) \to CF^-(\mathcal{H}_2, s)$

Juhász-Thurston-Zemke These maps are canonical, i.e.

- $\Phi(\mathcal{H}, \mathcal{H}) \simeq Id$
- $\Phi(\mathcal{H}_2, \mathcal{H}_3) \circ \Phi(\mathcal{H}_1, \mathcal{H}_2) \simeq \Phi(\mathcal{H}, \mathcal{H}_3)$ so invariant is natural.

③ What is its history?

Physics Equations	Yang-Mills	Seiberg-Witten
Count solns on W^4	Donaldson Invt	SW invt
Y; 3-mfd invt w/ cobordism maps • Look at eqns on $\mathbb{R}\times Y$ • Generators translation-invt solutions x, y on $\mathbb{R}\times Y$ • Differentials count solns on $\mathbb{R}\times Y$ approaching x, y as $t\to\pm\infty$. $\rightsquigarrow F(Y)$ • $\partial W = Y$; $F(W) = \left[\sum_x m(x)x\right] \in F(Y)$ solns on $W \cup ([0,\infty)\times Y)$ converging to x near cylindrical end	Instanton homology $I_*(Y)$ • Floer • Technically hard but many great applications	Seiberg-Witten Floer homology / Monopole Floer homology versions due to • Kronheimer-Mrowka • Manolescu
Cut $Y =$  U_β, Σ, U_α. Stretch metric along Σ $\rightsquigarrow$ eqns on $\mathbb{R}\times Y$ become $u: \mathbb{R}\times[0,1] \longrightarrow \mathcal{M}(\Sigma)$ where $\mathcal{M}(\Sigma)$ is solutions on $\mathbb{R}^2\times\Sigma$. • $\mathcal{M}(\Sigma)$ symplectic; u pseudoholo	$\updownarrow$ Atiyah-Floer Conj Symplectic instanton homology • $\mathcal{M}(\Sigma)$ Flat $SU(2)$ connections on Σ • L_α, L_β connections extending over U_α, U_β $HF(\mathcal{M}(\Sigma), L_\alpha, L_\beta)$	? Solns to vortex equations (SW eqns on $\mathbb{R}^2\times\Sigma$) are $\mathrm{Sym}^g(\Sigma)$. Lagns built from curves bounding disks in U_α, U_β.

Status of relationship

• S^1-equivariant SWFH $\longleftrightarrow$ Monopole Floer homology $\longleftrightarrow$ Heegaard Floer

Lidman-Manolescu

$$\begin{tikzcd} \text{Monopole Floer homology} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{Heegaard Floer} \arrow[dl, leftrightarrow] \\ & \text{ECH} & \end{tikzcd}$$

Kutluhan-Lee-Taubes; Colin-Ghiggini-Honda-Taubes

Disadvantages Less natural

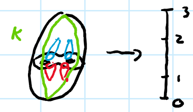
• Basepoint dependence
• Uniqueness of htpy

$$\begin{tikzcd} & \mathcal{H}_2 \arrow[dr] \arrow[d, phantom, "\Downarrow ? \Downarrow"] & \\ \mathcal{H}_1 \arrow[ur] \arrow[dr] & {} & \mathcal{H}_3 \\ & \mathcal{H}_4 \arrow[ur] & \end{tikzcd}$$

④ What are its advantages?

⓪ Trivial to list generators.

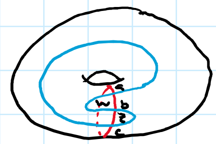
① Knot Theory and surgery formula

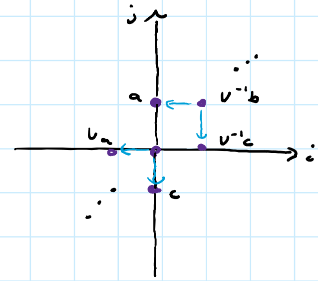


$K \subseteq S^3 \rightsquigarrow CFK^\infty(K)$

(Ozsváth-Szabó, Rasmussen)

- Free, finitely gen'd, graded, $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain cpx over $\mathbb{F}[U,U^{-1}]$

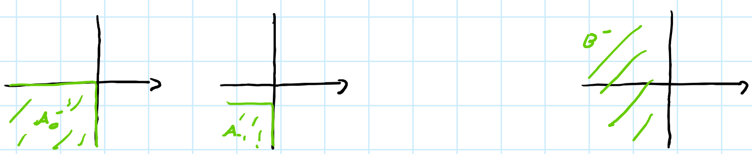




Thm (Ozsváth-Szabó) $CFK^\infty(K)$ determines $CF^-(S^3_{p/q}(K))$

How so? Focus on $n>0$, and $[0] \in \mathbb{Z}/n\mathbb{Z} \simeq \mathrm{Spin}^c(S^3_n(K))$ spin structure.

$A_s^- = F_{(0,s)} = C\{(i,j) : i \le 0, j \le s\}$ $\quad B_s^- = U F_{(0,s)} = C\{i,j\} : i \le 0\}$



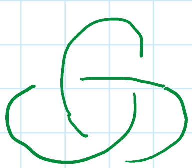
Large surgery For $n \ge g(K)$, $CF^-(S^3_n(K)) \simeq A_0^-[-d(L(n,1),[0])]$ $\nwarrow \frac{n-1}{4}$

Any surgery $CF^-(S^3_n(K)) \simeq \mathrm{Cone}\left(\bigoplus_i A_i^- \xrightarrow{D} \bigoplus_j B_j^-\right)$

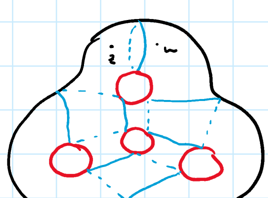
$$S^3_{+1}(K) \qquad \begin{tikzcd} A_{-2}^- \arrow[dr] & & A_{-1}^- \arrow[dl, "v"'] \arrow[dr, "h"] & & A_0^- \arrow[dl, "v"'] \arrow[dr, "h"] & & A_1^- \arrow[dl, "v"'] \arrow[dr, "h"] & & A_2^- \arrow[dl] \\ & B_{-1}^- & & B_0^- & & B_1^- & & B_2^- & \end{tikzcd}$$

- v inclusion
- h inclusion into $U F_{s,0} = \tilde{B}$, then a choice of flip homotopy F
- For CF^-, actually requires a completion.

Intuition

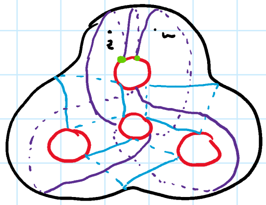


$\rightsquigarrow$



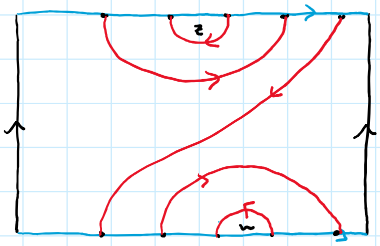
$\exists$ a Heegaard diagram for which one curve is a knot meridian

{ replace w/ surgery longitude



Each of the green intersection pts corresponds to a copy of the knot Floer complex.

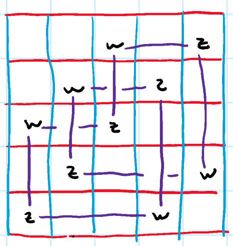
Exercise Compute $CFK^\infty(Y_1)$. Here is a Heegaard diagram:



② Combinatoriality

ⓐ Thm (Sarkar-Wang) Suppose that $\mathcal{H}$ is such that regions in $(\Sigma - \alpha - \beta)$ containing no basepoint are squares or bigons. ("nice")

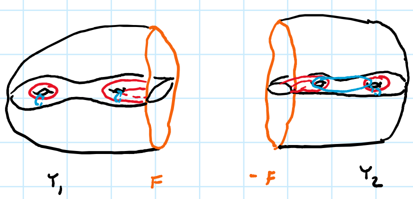
Then the Heegaard Floer differential is combinatorial and counts exactly embedded bigons & rectangles



Eg Every knot has a "grid diagram" on the torus.

ⓑ Combinatorial computation for Seifert fibred spaces (Némethi)

③ Cut-and-paste "bordered" methods $\widehat{CF}(Y) \simeq \widehat{CFA}(Y_1) \boxtimes_{\mathcal{A}(F)} \widehat{CFD}(Y_2)$



Why helpful?

① If $Y = U_1 \sqcup_{\varphi} U_2$ handlebodies, all information is in gluing map.

$F \rightsquigarrow \mathcal{A}(F)$

$Y \rightsquigarrow \widehat{CFA}(Y_1)$ A_∞-module

$\widehat{CFD}(Y_2)$ "Type D module"

Decompose $\varphi = \underbrace{\varphi_k \circ \dots \circ \varphi_1}$

std elements of mapping class group of surface

$$\rightsquigarrow \quad Y = \mathcal{H}_1 \sqcup M_{\varrho_1} \sqcup M_{\varrho_2} \sqcup \cdots \sqcup \mathcal{H}_2$$

$$\rightsquigarrow \quad \widehat{CF}(Y) \simeq \widehat{CFA}(U_1) \boxtimes \underbrace{\widehat{CFDA}(M_{\varrho_1})}_{\text{Precomputed}} \boxtimes \cdots \boxtimes \widehat{CFD}(U_2)$$

- Reduction possible by finding a minimal model at each stage before continuing.

② Many favorite constructions (Dehn surgery, satellites) use torus bdy, for which $\mathcal{A}(F)$ is a normal dga.