Math 549: Suggested Exercises for Lectures 18 and 19

Bump Chapter 12, Kirillov Sections 4.8-9

1. Let $V_k = S^k\mathbb{C}^2$ be the representation of $\mathfrak{sl}(2,\mathbb{C})$ constructed in class.
   (a) Show that V_2 is isomorphic to the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$.
   (b) Which of the representations V_k lift to representations of $SO(3,\mathbb{R}$?
2. Show that $\Lambda^n\mathbb{C}^n \simeq \mathbb{C}$ as a representation of $\mathfrak{sl}(n,\mathbb{C})$. Does this also work for $\mathfrak{gl}(n,\mathbb{C})$.
3. If (ρ, V) is a representation of $SL(2,\mathbb{R})$, $SU(2)$ or $SL(2,\mathbb{C})$, we can restrict the character function of ρ to the diagonal subgroup, obtaining a Laurent polynomial
$$\xi_\rho = \mathrm{tr}\rho\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}.$$
   (a) Compute $\xi_\rho(t)$ for the symmetric power representations. Show that the resulting polynomials are independent and determine the representation ρ.
   (b) Show that $\xi_{\rho\otimes\rho'} = \xi_\rho\xi\rho'$. Use this to determine the decomposition of products of the symmetric power representations into irreducibles.
4. Let V be a representation of $\mathfrak{sl}(2,\mathbb{C})$, and let $C \in \mathrm{End}(V)$ such that $C(v)$ is
$$EFv + FEv + \frac{1}{2}H^2v.$$
   (a) Show that C commutes with the action of $\mathfrak{sl}(2,\mathbb{C})$, that is, show that for $x \in \mathfrak{sl}(2,\mathbb{C})$, then $[\rho(x), C] = 0$. (Hint: Jacobi Identity.)
   (b) Show that if $V = V_k$ is the irreducible representation with highest weight k, then $C = c_k\mathrm{Id}$. Compute the constant c_k.
   (c) Show that the isomorphism $\mathfrak{so}(3,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})$ identifies C with a multiple of $\rho(J_x)^2 + \rho(J_y)^2 + \rho(J_z)^2$.

   This is a special case of the *Casimir element*, which we'll discuss more shortly.
5. Complete the computation from lecture (or Kirillov Section 4.9) to find the eigenvalues and multiplicities of the operator
$$H = -\Delta - \frac{c}{r}, \qquad c > 0$$
   in $L^2(\mathbb{R}^3, \mathbb{C})$.