Math 549: Suggested Exercises for Lectures 16 and 17

Bump Chapters 3 and 4, Kirillov Section 4.7

1. Suppose that T is a bounded operator on the Hilbert space H, and suppose that for each $\epsilon > 0$ there exists a compact operator such that $T - T_\epsilon| < \epsilon$. Prove that T is compact.

2. *Hilbert-Schmidt operators* Let X be a locally compact Hausdorff space with a positive Borel measure μ. Assume that $L^2(X)$ has a countable basis. Let $K \in L^2(X \times X)$. Consider the operator on $L^2(X)$ with kernel K defined by

   $$Tf(x) = \int_X K(x, y)f(y)d\mu(y).$$

   Let ϕ_i be an orthonormal basis for $L^2(X)$. Expand K in a Fourier expansion:

   $$K(x, y) = \Sigma_{i=1}^{\infty}\psi_i(x)\overline{\phi_i(y)}$$

   where $\psi_i = T\phi_i$. Show that $\Sigma|\psi_i|^2 = \int\int |K(x, y)|^2 d\mu(x)d\mu(y) < \infty$. Consider the operator T_N with kernel

   $$K_N(x, y) = \Sigma_{i=1}^{N}\psi_i(x)\overline{\phi_i(y)}.$$

   Show that T_N is compact, and deduce that T is compact.

3. Be sure to Exercises 6 & 7 from last week, which deal with the Peter-Weyl Theorem.

4. *Groups with two compact topologies* This exercise is provided in response to a question asked in class. Let $\{p_n\}$ be some ordering of the prime integers. Consider the inverse system $\{S^1_n, \alpha_n\}$ where for each n S^1_n is a copy of the circle S^1 and $\alpha_n : S^1_n \to S^1_{n-1}$ is $\alpha_n(x) = p_n x$. Check that the inverse limit is isomorphic to S^1 as groups but not as topological spaces.

   However, check out T. Stewart, "Uniqueness of the Topology in Certain Compact Groups" for a uniqueness result in the case of groups with disconnected center.