Math 549: Suggested Exercises for Lectures 14 and 15

Kirillov Sections 4.5-7, Bump Chapter 2 and 15

1. Let (ρ, V) be an irreducible complex representation of a compact group, and let $G \times G$ act on the space M_ρ by $(g,h)\phi(x) = \phi(g^{-1}xh)$. Show that $M_\rho \simeq V^* \otimes V$ as $(G \times G)$-modules.

2. Let G be a compact group and $g, h \in G$. Show that g and h are conjugate if and only if $\chi(g) = \chi(h)$ for every irreducible character χ. Show also that every character is real-valued if and only if every element is conjugate to its inverse.

3. Let G be a compact group and $C(G)$ be the space of continuous functions on G, with the convolution

$$f_1 * f_2(g) = \int_G f_1(gh^{-1}) f_2(h) dh = \int_G f_1(h) f_2(h^{-1}g) dh.$$

   Prove this operation is associative, so that $C(G)$ is a ring with respect to convolution. If (ρ, V) is an irreducible represenation of G, prove that M_ρ is a 2-sided ideal in $C(G)$.

4. For a representation of a Lie algebra $\mathfrak{g}$, define the space of *coinvariants* by $V_{\mathfrak{g}} = V/\mathfrak{g}V$, where $\mathfrak{g}V$ is the subspace spanned by xv for $x \in \mathfrak{g}$ and $v \in V$.

   (a) Show that if V is completely reducible, the composition $V^{\mathfrak{g}} \hookrightarrow V \to V_{\mathfrak{g}}$ is an isomorphism.

   (b) Give an example to show that this is not true in general.

5. Compute the dimensions of the flag manifolds for $\mathfrak{su}(n)$, $\mathfrak{sp}(2n)$, and $\mathfrak{so}(n)$.

6. Let G be totally disconnected, and let $\rho : G \to \mathrm{GL}(n, \mathbb{C})$ be a finite-dimensional representation. Show that the kernel of ρ is open. Conclude that if $\mathbb{Z}_p$ is the ring of p-adic integers, the compact group $\mathrm{GL}(n, \mathbb{Z}_p)$ has no faithful finite-dimensional representation.

7. Let G be a compact abelian group and $H \subset G$ a closed subgroup. Let $\chi : H \to \mathbb{C}^x$ be a character. Show that χ can be extended to a character of G. (Hint: Apply the second part of the Peter-Weyl theorem to the space $V = \{f \in L^2(G) : f(hg) = \chi(h)f(g)\}$. We know V is nonzero because if $\phi \in C(G)$, then $f(g) = \int \phi(hg)\chi(h)^{-1} dh$ is an element of V, and we can use Urysohn's lemma to construct a ϕ for which f is nonzero.)