Math 549: Suggested Exercises for Lectures 12 and 13

Kirillov Sections 3.6, 3.9, 4.1-4, Bump Chapters 1 and 2

1. Let $SO(p, q)$ be the special indefinite orthogonal group, that is, the set of transformations preserving a nondegenerate symmetric bilinear form of signature (p, q). Prove that the complexification of $\mathfrak{so}(p, q)$ is $\mathfrak{so}(p + q)$.

2. Let G be a complex connected simply-connected Lie group, with Lie algebra $\mathfrak{g}$, and let $\mathfrak{k} \subset \mathfrak{g}$ be a real form of $\mathfrak{g}$.

   (a) Define the $\mathbb{R}$-linear map $\Theta : \mathfrak{g} \to \mathfrak{g}$ by $\Theta(x + iy) = x - iy$, $x, y \in \mathfrak{k}$. Show that Θ is an automorphism of $\mathfrak{g}$ (considered as a real Lie algebra) which can be lifted uniquely to an automorphism $\Theta : G \to G$ (considered as a real Lie group.

   (b) Let $K = G^{\Theta}$ be the fixed set of Θ. Show that K is a real Lie group with Lie algebra $\mathfrak{k}$.

3. (a) Let V and W be irreducible representations of a Lie group G. Show that $(V \otimes W^*)^G = 0$ if V is not isomorphic to W, and that $(V \otimes V^*)^G$ is canonically isomorphic to $\mathbb{C}$.

   (b) Let V be an irreducible representation of a Lie algebra $\mathfrak{g}$. Show that V^* is also irreducible, and deduce from this that the space of $\mathfrak{g}$-invariant bilinear forms on V has dimension zero or one.

4. Let G be a complex connected Lie group (that is, a Lie group with the structure of a complex manifold such that multiplication and inversion are analytic maps).

   (a) Show that $g \mapsto \mathrm{Ad}_g$ is an analytic map $G \to \mathfrak{gl}(\mathfrak{g})$.

   (b) Assume that G is compact. Show that $\mathrm{Ad}_g = \mathrm{Id}$ for any $g \in G$.

   (c) Prove that a connected compact complex Lie group is always commutative.

5. Let $\mathfrak{g}$ be a Lie algebra and $(,)$ a symmetric bilinear form on $\mathfrak{d}$ which is invariant under ad. Show that the element $\omega \in ((g)^*)^{\otimes 3}$ given by

$$\omega(x, y, z) = ([x, y], z)$$

   is skew-symmetric and ad-invariant.

6. Let $G = SU(2) \simeq S^3$.

   (a) Let ω be a left-invariant 3-form whose value at $\mathrm{Id} \in G$ is defined by

$$\omega(x_1, x_2, x_3) = \mathrm{tr}([x_1, x_2]x_3)$$

   for $x_i \in \mathfrak{g}$. Show that ω is $\pm 4dV$ where dV is the standard volume form on S^3. (Hint: Let x_1, x_2, x_3 be some basis in $\mathfrak{su}(2)$ orthonormal with respect to $\frac{1}{2}(a\bar{b}^t))$.

   (b) Show that $\omega' = \left(\frac{1}{8\pi^2}\right)\omega$ is a bi-invariant form on G such that for appropriate choice of orientation on G, we have $\int_G \omega' = 1$.