1. Verify that a connected Lie group is always generated by a neighborhood $U$ of the identity.

2. Suppose that $G$ is a connected Lie group and $H$ is any Lie group. If $\Phi, \Psi$ are Lie group homomorphisms such that $\Phi_* = \Psi_* : \mathfrak{g} \to \mathfrak{h}$, show that $\Phi = \Psi$.

3. Consider the basis for $\mathfrak{so}(3)$ given by

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. $$

To what subgroups in $SO(3)$ do these elements correspond? Check that the exponentiation of $tJ_x$, $tJ_y$, and $tJ_z$ generate a neighborhood of the identity, hence the group.

4. Consider the basis for $\mathfrak{su}(2)$ given by

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. $$

Determine the isomorphism $\mathfrak{su}(2) \to \mathfrak{so}(3)$ induced by the twofold covering map $SU(2) \to SO(3)$.

5. (a) Use Gram-Schmidt to show every matrix in $SL(n, \mathbb{C})$ can be uniquely expressed as $A = BC$, where $B \in SU(n)$ and $C$ is in the subgroup of $SL(n, \mathbb{C})$ consisting of upper-triangular matrices with positive real entries on the diagonal.

(b) Confirm that $SL(2, \mathbb{C})$ is diffeomorphic to $S^3 \times \mathbb{R}^3$ and thus simply connected.

6. (a) Use Gram-Schmidt to show that every matrix in $SL(n, \mathbb{R})$ can be uniquely expressed as $A = BC$, where $B \in SO(n)$ and $C$ is in the subgroup of $SL(n, \mathbb{R})$ consisting of upper triangular matrices with positive entries on the diagonal.

(b) Show that $SL(2, \mathbb{R})$ is diffeomorphic (not Lie group isomorphic) to $S^1 \times \mathbb{R}^2$.

(c) Show that the universal cover of $SL(2, \mathbb{R})$ has infinite cyclic center.

7. (a) Prove that any matrix in $SL(2, \mathbb{R})$ is either:

- i. hyperbolic, with eigenvalues $\lambda_1 \neq \lambda_2 \in \mathbb{R}$
- ii. parabolic, with eigenvalues $\lambda_1 = \lambda_2 = \pm 1$
- iii. elliptic, with eigenvalues $\lambda_1 = \overline{\lambda_2} \in U(1)$ \{±1\}.

(b) Show that the image of the exponential map in $SL(2, \mathbb{R})$ is exactly the set of matrices $A$ for which $tr(A) > 0$, $A$ is elliptic, or $A = -I$. 

8. Let $G$ be the universal covering group of $SL(2, \mathbb{R})$. Show there is no faithful representation of $G$, that is, no injective homomorphism $\rho : G \to GL(n, \mathbb{R})$, as follows:

(a) Suppose such a $\rho$ exists and let $\rho_* : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ be the induced Lie algebra homomorphism. Show that $\phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ given by $\phi(A + iB) = \rho_*(A) + i\rho_*(B)$ is also a Lie algebra homomorphism.

(b) Show there is a Lie group homomorphism $\Phi : SL(2, \mathbb{C}) \to GL(n, \mathbb{C})$ with $\Phi_* = \phi$.

(c) Write down a commutative diagram involving all five groups mentioned so far, and use it to argue that $\rho$ is not injective.

9. Persons who have previous familiarity with quantum mechanics are encouraged to read through Exercise 3.14 in Kirillov.