Math 549: Suggested Exercises for Lectures 10 and 11

Kirillov Chapter 3, Bump Chapter 8, 14 and 15, Lee Chapter 14

1. Verify that a connected Lie group is always generated by a neighborhood U of the identity.

2. Suppose that G is a connected Lie group and H is any Lie group. If Φ, Ψ are Lie group homomorphisms such that $\Phi_* = \Psi_* : \mathfrak{g} \to \mathfrak{h}$, show that $\Phi = \Psi$.

3. Consider the basis for $\mathfrak{so}(3)$ given by

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   To what subgroups in $SO(3)$ do these elements correspond? Check that the exponentiation of tJ_x, tJ_y, and tJ_z generate a neighborhood of the identity, hence the group.

4. Consider the basis for $\mathfrak{su}(2)$ given by

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

   Determine the isomorphism $\mathfrak{su}(2) \to \mathfrak{so}(3)$ induced by the twofold covering map $SU(2) \to SO(3)$.

5. (a) Use Gram-Schmidt to show every matrix in $SL(n, \mathbb{C})$ can be uniquely expressed as $A = BC$, where $B \in SU(n)$ and C is in the subgroup of $SL(n, \mathbb{C})$ consisting of upper-triangular matrices with positive real entries on the diagonal.
   (b) Confirm that $SL(2, \mathbb{C})$ is diffeomorphic to $S^3 \times \mathbb{R}^3$ and thus simply connected.

6. (a) Use Gram-Schmidt to show that every matrix in $SL(n, \mathbb{R})$ can be uniquely expressed as $A = BC$, where $B \in SO(n)$ and C is in the subgroup of $SL(n, \mathbb{R})$ consisting of upper triangular matrices with positive entries on the diagonal.
   (b) Show that $SL(2, \mathbb{R})$ is diffeomorphic (not Lie group isomorphic) to $S^1 \times \mathbb{R}^2$.
   (c) Show that the universal cover of $SL(2, \mathbb{R})$ has infinite cyclic center.

7. (a) Prove that any matrix in $SL(2, \mathbb{R})$ is either:
      i. hyperbolic, with eigenvalues $\lambda_1 \neq \lambda_2 \in \mathbb{R}$
      ii. parabolic, with eigenvalues $\lambda_1 = \lambda_2 = \pm 1$
      iii. elliptic, with eigenvalues $\lambda_1 = \overline{\lambda_2} \in U(1)\ \{\pm 1\}$.
   (b) Show that the image of the exponential map in $SL(2, \mathbb{R})$ is exactly the set of matrices A for which $tr(A) > 0$, A is elliptic, or $A = -I$.

8. Let G be the universal covering group of $SL(2,\mathbb{R})$. Show there is no faithful representation of G, that is, no injective homomorphism $\rho : G \to GL(n,\mathbb{R})$, as follows:

   (a) Suppose such a ρ exists and let $\rho_* : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{gl}(n,\mathbb{R})$ be the induced Lie algebra homomorphism. Show that $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$ given by $\phi(A+iB) = \rho_*(A) + i\rho_*(B)$ is also a Lie algebra homomorphism.

   (b) Show there is a Lie group homomorphism $\Phi : SL(2,\mathbb{C}) \to GL(n,\mathbb{C})$ with $\Phi_* = \phi$.

   (c) Write down a commutative diagram involving all five groups mentioned so far, and use it to argue that ρ is not injective.

9. Persons who have previous familiarity with quantum mechanics are encouraged to read through Exercise 3.14 in Kirillov.