Math 549: Suggested Exercises for Lectures 8 and 9

References: Lee Chapter 7, Kirillov Sections 2.4-5, 3.1-3, Bump Chapters 6-7, 13.

1. Show there is an embedding $\mathbb{Z}/n\mathbb{Z} \subset SU(n) \times U(1)$ such that $(SU(n) \times U(1))/\mathbb{Z}/n\mathbb{Z} \simeq U(n)$.

2. Let $\mathrm{Fl}_k(\mathbb{C}^n)$ be the set of k-flags in $\mathbb{C}^n$, that is, ordered k-tuples of mutually orthogonal one-dimensional complex subspaces of $\mathbb{C}^n$. Show that $U(n)$ acts transitively on $\mathrm{Fl}_k(\mathbb{C}^n)$ and determine the stabilizer of $(\langle e_1 \rangle, \langle e_2 \rangle, \ldots, \langle e_k \rangle)$. With what homogeneous space can $\mathrm{Fl}_k(\mathbb{C}^n)$ be identified? What is its dimension? Show that $\mathrm{Fl}_k(\mathbb{C}^n)$ admits a smooth surjection to $\mathrm{Gr}_k(\mathbb{C}^n)$. What are the fibres of this map?

3. Show that there are exactly two 2-dimensional Lie algebras up to isomorphism. Describe each of them as a subspace of $\mathfrak{gl}(2, \mathbb{R})$.

4. Let G be a Lie group.

   (a) Compute the derivative at (e, e) of the multiplication $m : G \times G \to G$.

   (b) Compute the derivative at e of inversion $i : G \to G$.

   (c) Show that G is abelian if and only if i is a homomorphism. (Hint: This is easy.)

   (d) Show that if G is abelian, then its Lie algebra $\mathfrak{g}$ has trivial bracket.

   (e) Give a counterexample to the converse of (d).

5. If H is a Lie group and $f : G \to H$ is a connected covering space (so that G is also a Lie group), prove that the induced map $f_* : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism of Lie algebras.

6. Prove that a normal discrete subgroup of a Lie group is always central. Use this to give an alternate proof that the fundamental group of a connected Lie group is abelian.

7. (Bump functions, for those who haven't seen them before) Check that:

   (a) The following function $f : \mathbb{R} \to R$ is smooth:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   (b) There exists a smooth function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) > 0$ if $x \in (-1, 1)$ but $g(x) = 0$ otherwise.

   (c) There exists a smooth function $h : \mathbb{R} \to [0, 1]$ such that $h(x) = 0$ if $x \leq 0$ but $h(x) = 1$ if $x \geq 1$.

   (d) For any $\epsilon > 0$, there exists a smooth $\psi : \mathbb{R}^n \to [0, 1]$ such that $\phi(x) = 0$ if $|x| \geq 2\epsilon$ but $\phi(x) = 1$ if $|x| \leq \epsilon$.

   (e) If M is a smooth manifold, $p \in U$ open in M, there exists a smooth $\phi : M \to \mathbb{R}$ such that $\phi(x) = 0$ if $x \notin U$ but $\phi(x) = 1$ on some neighborhood V of p.