Math 549: Suggested Exercises for Lectures 6 and 7

References: Kirillov Sections 2.1-5, Bump Section 5, Lee Chapter 7.

1. Let $f : G \to H$ be a Lie group homomorphism which is also a diffeomorphism. Prove that f^{-1} is also a Lie group homomorphism.

2. Prove that the fundamental group of a (topological) manifold is always countable.

3. If $p : M \to N$ is a smooth map of manifolds with M compact, N connected, and $D_x f$ an isomorphism for all $x \in M$, prove that p is a covering map. Give a counterexample to show that this is not true if M is not compact.

4. Prove that $\mathbb{CP}^1$ is diffeomorphic to S^2 by expressing both as homogeneous spaces.

5. If G is a Lie group acting smoothly (but not necessarily properly) on a manifold M, show that the stabilizer group $G_x = \{g \in G : gx = x\}$ of any $x \in M$ is a closed embedded Lie subgroup.

6. If $H \subset G$ is a closed embedded Lie subgroup which is normal in G, show that G/H is a Lie group and the projection $\pi : G \to G/H$ is a group homomorphism.

7. If $\phi : G_1 \to G_2$ is a smooth homomorphism of Lie groups (not necessarily proper), show that $\ker(\phi)$ is a closed embedded Lie subgroup. If ϕ is also surjective, show there is an isomorphism of Lie groups $G_2 \simeq G_1 / \ker(\phi)$.

8. Prove that an injective Lie group homomorphism of compact connected Lie groups of the same dimension is necessarily an isomorphism.