Math 549: Suggested Exercises for Lectures 4 and 5

References: For McKay graphs, `http://www.math.lsa.umich.edu/~idolga/McKaybook.pdf`. For differential geometry, Spivak Chapters 1 and 2.

1. For G finite and a representation ρ of G over $\mathbb{C}$, prove that the McKay graph of G is connected if and only if $\rho : G \to GL(V)$ is injective. (That is, the representation is *faithful*.)

2. Prove that the canonical representation of a finite subgroup of $SU(2)$ is always self-dual.

3. Finish the computation of the character table of the binary tetrahedral group started in class, and use your computation to confirm the McKay graph. (Use the description of the group in terms of matrices in $SU(2)$ at `http://www.math.stonybrook.edu/~tony/bintet/quat-rep.html`.

4. Let $\widetilde{\mathbb{R}}$ be the smooth structure on the real line equipped with one chart $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(x) = x^3$. Prove this is not equal to the usual smooth structure on $\mathbb{R}$, but *is* equivalent to the usual smooth structure under diffeomorphism.

5. Prove the chain rule stated in class: If M, N, P are manifolds and $M \xrightarrow{f} N \xrightarrow{g} P$ are smooth functions, then $(f \circ g)_* = f_* \circ g_*$.

6. If $f : M \to N$ is an injective immersion of manifolds with M compact, prove that N is an embedding.

7. Show that any regular k-dimensional submanifold of a k-dimensional manifold must be an open subset. (Hint: Start by showing that any submersion is an open map.)

8. Regard $\mathbb{RP}^2$ as the quotient space of S^2 by identifying antipodal points. Define $F : \mathbb{R}^3 \to \mathbb{R}^4$ by $F(x, y, z) = (x^2 - y^2, xy, xz, yz)$. This descends to a map $\widetilde{F} : \mathbb{RP}^2 \to \mathbb{R}^4$. Prove this map is an embedding.

9. Let $S(n)$ be the space of symmetric $n \times n$ real matrices.

   (a) Show that $S(n)$ is a real vector space of dimension $\frac{n(n+1)}{2}$, and that the map $f : \mathrm{GL}(\mathrm{n}, \mathbb{R}) \to S(n)$ given by $f(A) = A^T A$ is a submersion.

   (b) Show that $O(n) = \{A \in GL(n, \mathbb{R}) : A^T A = I\}$ is a compact Lie group of dimension $\frac{n(n-1)}{2}$.

10. For S a subset of a group G, let $S^2 = \{gh : g, h \in S\}$. If U is a neighborhood of the identity e in a Lie group G, show there is another neighborhood V of e such that $V^2 \subset U$.