Math 549: Suggested Exercises for Lectures 20 and 21

References: Mitchell's "Notes on Principal Bundles"

1. Confirm that if a Lie group G acts on a manifold M smoothly, freely, and properly, then $M \to M/G$ is a principal G-bundle.

2. Let G and H be topological groups. Confirm that if X is a right G-space, Y is a (G, H)-space, and Z is a left H-space, then $(X \times_G Y) \times_H Z$ is homeomorphic to $X \times_G (Y \times_H Z)$.

3. Let $\pi : P \to B$ be a principal G-bundle, and $f : B' \to B$ a continuous map. Check that bundle maps $Q \to f^*P$ are in one-to-one correspondence with commutative diagrams

$$\begin{array}{ccc} Q & \longrightarrow & P \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

such that the top row is an equivariant map.

4. Show that any fibre bundle $E \to B$ with structure group G is isomorphic to $P \times_G F \to P \times_G * = B$ for some principal G-bundle $P \to B$ and left action of G on F.

5. Let γ be the canonical line bundle over $\mathbb{RP}^n$. Prove that $\gamma^{\oplus(n+1)}$ is isomorphic to the Whitney sum of $T\mathbb{RP}^n$ with a trivial line bundle.

6. Prove that a Serre fibration with weakly contractible fiber admits a section.

7. Classify the real line bundles over a CW complex B.

8. Determine EG and BG for $G = \mathbb{Z}$ and $G = \mathbb{Z}/p\mathbb{Z}$.