Propn $F : M \to N$ a diffeomorphisms $\Rightarrow F[X,Y] = [FX, FY]$

Proof For $g \in C^\infty(N)$,

$$
\begin{aligned}
[FX, FY]g &= (FX)(FY)g - (FY)(FX)g \\
&= FX\left(Y(g\circ F)\circ F^{-1}\right) - FY\left(X(g\circ F)\circ F^{-1}\right) \\
&= X\left(Y(g\circ F)\circ F^{-1}\circ F\right)\circ F^{-1} - Y\left(X\circ(g\circ F)\circ F^{-1}\circ F\right)\circ F^{-1} \\
&= XY(g\circ F)\circ F^{-1} - YX(g\circ F)\circ F^{-1} \\
&= [X,Y](g\circ F)\circ F^{-1} \\
&= F[X,Y]g
\end{aligned}
$$

(*)

Lemma As a GD, $FY(g) = Y(g\circ F)\circ F^{-1}$.

Pf

$$
\begin{aligned}
(FY)(g)(F(p)) &= FY(F(p))g \\
&= F_* Y(p)g \\
&= Y(p)(g\circ F) \\
&= Y(g\circ F)(p) \\
&= Y(g\circ F)\left(F^{-1}(F(p))\right)
\end{aligned}
$$

Defn A Lie algebra is a real vector space equipped w/ a bilinear operation $X, Y \mapsto [X,Y]$ satisfying

1. Antisymmetry $[X,Y] = -[Y,X]$
2. Jacobi identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

Lie algebras are not algebras.

Example 0 Any vector space w/ $[v_1, v_2] = 0$.

Example 1 Any algebra w/ $[x,y] = xy - yx$.

$[[x,y],z] + [[y,z]x] + [[z,x],y] = 0$ after writing out twelve cancelling terms

Example 2 $VF(M)$ on any smooth M. The Jacobi identity follows from the fact that XYf is a sensible term even if XY isn't.

Example 3 Any subspace of a Lie algebra closed under the Lie bracket.

$\{A \in M_n(\mathbb{R}) : tr(A) = 0\}$ $tr(AB - BA) = tr(AB) - tr(BA) = 0$.

Example 4 G a Lie group, $L_g : G \to G$, $h \mapsto gh$. Say $X \in VF(G)$ is left-invariant

if $\forall\ g \in G$, $L_g X = X$. Let $Lie(G) := \left\{ \begin{array}{l} \text{Left-invt vector} \\ \text{fields on } G \end{array} \right\}$.

Propn $Lie(G)$ is closed under $[\,,\,]$

PF $L_g[X,Y] = [L_g X, L_g Y]$
$= [X,Y]$

Example $G = S^1$

Left invt vector field has same magnitude & orientation at each point.

$\exists$ a canonical linear map $\mathrm{Lie}(G) \longrightarrow T_e G$

$$X \longmapsto X(e)$$

Propn This is an isomorphism of vector spaces.

Pf Say X left-invt and $X(e) = 0$. $\forall g \in G$, $X(g) = (L_g)_*(X(e)) = (L_g)_*(0) = 0$ so the map is injective. Conversely given $v \in T_e(G)$, consider the (cts) map $X(g) = (L_g)_*(v)$. To show our map is surjective, suffices to check this is smooth. Let $\gamma : (-\varepsilon, \varepsilon) \to G$ st $v = \gamma'(0)$, so that $vF = (F \circ \gamma)'(0)$.
Then if $x_i : U \to \mathbb{R}$ are the coordinate ftns on a chart,
$X(g)(x_i) = (L_g)_* v(x_i) = v(x_i \circ L_g) = (x_i \circ L_g \circ \gamma)'(0)$. Want to show this is a smooth map $U \to \mathbb{R}$. But if $m : G \times G \to G$ is multiplication,
$(x_i \circ L_g \circ \gamma)(t) = x_i \circ m(g, \gamma(t))$. This is a smooth ftn $F : U \times (-\varepsilon, \varepsilon) \to \mathbb{R}$, so
$\frac{\partial F}{\partial t}(g, 0) = (x_i \circ L_g \circ \gamma)'(0)$ is a smooth ftn of g. $\square$

Let $\mathfrak{g}, \mathfrak{h}$ be Lie algebras. A <u>Lie algebra homomorphism</u> is a linear map $f : \mathfrak{g} \to \mathfrak{h}$ st $\forall\, x, y \in \mathfrak{g}$, $f([x,y]) = [f(x), f(y)]$. An isomorphism which is also a Lie algrebra homomorphism is a <u>Lie algebra isomorphism</u>.

<u>Exercise</u> The inverse of a Lie algebra isomorphism is also a Lie algebra isomorphism.

Propn The natural map $Lie(GL(n,\mathbb{R})) \to M_{n\times n}(\mathbb{R}) = T_e GL(n,\mathbb{R})$ (w/ brackets of left-invt vector fields and $AB-BA$ respectively) is a Lie algebra isomorphism.

Henceforth this Lie algebra is $gl(n,\mathbb{R})$.

PF Certainly this is an isomorphism of vector spaces. So it suffices to consider the bracket. Let $x_{ij} : GL(n,\mathbb{R}) \to \mathbb{R}$ be the coordinat functions.

Let $U = X(e)$, $V = Y(e) \in T_e GL(n,\mathbb{R}) = M_n(\mathbb{R})$.

For $g \in GL(n,\mathbb{R})$,
$$Y(x_{ij})_g = L_{g*}(V(x_{ij}))$$
$$= V(x_{ij} \circ L_g)$$
$$= V\left(\sum_{k=1}^{n} x_{ik}(g)\, x_{kj}\right)$$
$$= \sum_{k=1}^{n} x_{ik}(g)\, V(x_{kj})$$

So
$$Y(x_{ij}) = \sum_{k=1}^{n} (V x_{kj}) \cdot x_{ik}$$
$$= \sum_{k=1}^{n} \underbrace{v_{kj}}_{(kj)\text{th entry of matrix}} x_{ik}$$

Likewise $X(x_{ij}) = \sum_{\ell} u_{\ell j}\, x_{i\ell}$, so
$$XY(x_{ij}) = X\left(\sum_k v_{kj} \cdot x_{ik}\right)$$
$$= \sum_k v_{kj}\, X(x_{ik})$$
$$= \sum_{k,\ell} v_{kj} \cdot u_{\ell k}\, x_{i\ell}$$

Evaluate at $e = I$ where $x_{i\ell} = \delta_{i\ell}$. Then
$$XY(x_{ij})(e) = \sum_{k,\ell} v_{kj} \cdot u_{\ell k}\, \delta_{i\ell}$$
$$= \sum_{k\ell} v_{kj} u_{ik}$$
$$= (UV)_{ij}$$

Likewise $YX(x_{ij})_e = (VU)_{ij}$, so $[X,Y](e)(x_{ij}) = (UV - VU)x_{ij}$

$= [U,V]x_{ij}$

$= [X(e), Y(e)](x_{ij})$

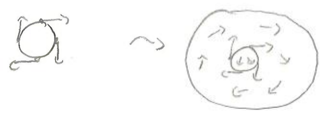
Since the x_{ij} span $T_e^*(GL(n,\mathbb{R}))$, $[X,Y] = [X(e), Y(e)]$.

Thm Let $F: G \to H$ be a Lie group homomorphism. Then the induced map $F_* = D_eF: T_eG \to T_eH$ is a Lie algebra homomorphism.

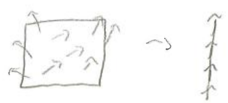
Defn If $F: M \to N$ smooth, $X \in VF(M)$ and $X' \in VF(N)$ are F-related if $\forall\, p \in M$, $X'(F(p)) = F_*(X(p))$.

Equivalently $\forall\, g \in C^\infty(N)$, $(X'g)\circ F = X(g\circ F)$.

Not unique in general



$X \not\Rightarrow X'$



$X' \not\Rightarrow X$

Lemma 1 X F-related to X', Y F-related to Y' $\Rightarrow [X,Y]$ is F-related to $[X',Y']$

Pf $YX(g\circ F) = Y(X'g\circ F) = Y'X'g\circ F$, and likewise $XY(g\circ F)$.

Lemma 2 $F: G \to H$ Lie group homomorphism, $X \in \mathfrak{g}$, $X' \in \mathfrak{h}$, $U = X(e)$, $U' = X'(e)$, $U' = F_* U$ $\Rightarrow$ X' is F-related to X.

Pf

$$\begin{tikzcd} G \arrow[r, "L_g"] \arrow[d, "F"'] \arrow[dr, phantom, "G"] & G \arrow[d, "F"] \\ H \arrow[r, "L_{F(g)}"'] & H \end{tikzcd}$$

$F_*(L_{g*})U = (L_{F(g)})_* F_* U$

$\Rightarrow F_* X(g) = X'(F(g))$

Pf of thm Given $U, V \in T_e G$, let $X, Y \in \mathfrak{g}$ be such that $X(e) = U$, $Y(e) = V$. Also let $U' = F_* U$, $V' = F_* V$, and $X', Y' \in \mathfrak{h}$ be such that $X'(e) = U'$, $Y'(e) = V'$. X is F-related to X' and Y is F-related to Y' $\Rightarrow$ $[X, Y]$ F-related to $[X', Y']$. So push-forward preserves the bracket.

Defn A Lie subalgebra is the image of an injective Lie algebra homomorphism, or a subspace closed under the Lie bracket.

Example $G \subseteq H$ immersed $\Rightarrow$ $\mathfrak{g} \subseteq \mathfrak{h}$ a Lie subalgebra.

Notation For the classical groups we write $\mathfrak{sl}(n, \mathbb{R}) \subseteq \mathfrak{gl}(n, \mathbb{R})$ and so on.

Example $SL(n, \mathbb{R}) = \ker(\det : GL(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\})$

$$\Rightarrow \mathfrak{sl}_n(\mathbb{R}) = T_I SL(n, \mathbb{R}) = \ker(D_I \det : (\underset{\substack{\| \\ M_{n \times n}(\mathbb{R})}}{T_I GL(n, \mathbb{R})} \to \underset{\substack{\| \\ \mathbb{R}}}{T_I \mathbb{R}})$$

For $A \in \mathfrak{gl}(n, \mathbb{R})$, let $\gamma(t) = I + tA$, then $\det \gamma(t) = \det(I + tA)$

$$= 1 + t(\operatorname{tr} A) + O(t^2)$$

$\Rightarrow (\det \circ \gamma)'(0) = \operatorname{tr} A$ $\Rightarrow$ $D_I \det = \operatorname{tr}$.

So $sl(n, \mathbb{R}) = \{\text{trace-free matrices in } gl(n, \mathbb{R})\}$

Ex $sl(2, \mathbb{R}) = \left\langle \underset{E}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{F}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{H}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \right\rangle$

$[H, E] = 2E$
$[H, F] = -2F$
$[E, F] = H$

Application There is no immersion of Lie groups $(\mathbb{R}^2, +) \hookrightarrow SL(2, \mathbb{R})$.

PF If there were, then $\text{Lie}(\mathbb{R}^2) \subseteq sl(2, \mathbb{R})$. But $\text{Lie}(\mathbb{R}^2) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$ w/ bracket $= 0$ since mixed partials of smooth functions commute. However, $Z(sl(2, \mathbb{R}))$ is only 1-dim'l.

Group	Dim	Lie algebra
$GL(n, \mathbb{R})$	n^2	$gl(n, \mathbb{R})$
$SL(n, \mathbb{R})$	$n^2 - 1$	$sl(n, \mathbb{R})$
$O(n)$, $SO(n)$	$\frac{n(n-1)}{2}$	$so(n)$ trace-free antisymmetric matrices
$U(n)$	n^2	$u(n)$ anti-hermitian matrices Diagonal $su(n) + \ell$
$SU(n)$	$n^2 - 1$	$su(n)$ trace-free antihermitian matrices
$Sp(n)$	$2n^2 + n$	$sp(n)$ quaternionic antihermitian matrices

Another perspective on identifying these tangent spaces

Exponentiation of matrices

$A \in M_n(\mathbb{R})$ or $M_n(\mathbb{C})$

$$\exp(A) = I + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots = \sum_{k \geq 0} \frac{A^k}{k!}$$

Converges uniformly if $|a_{ij}| \leq c$ $\forall\, i,j$ and some C, since this bounds the coefficients of $\frac{A^k}{k!}$, $\left|\left(\frac{A^k}{k!}\right)_{ij}\right| \leq \frac{n(c)^k}{k!}$.

Properties

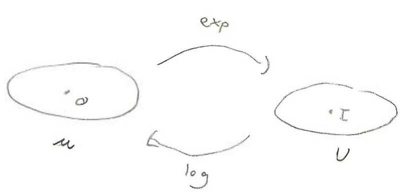
① $\exp(BAB^{-1}) = \sum \frac{BA^kB^{-1}}{k!} = B\exp(A)B^{-1}$

② $\exp(A+B) = \exp(A)\exp(B)$ if A & B commute

③ $\exp(-A) = \exp(A)^{-1}$

④ $\exp((t+s)A) = \exp(tA)\exp(sA)$

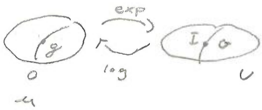
$\exp : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$ is clearly smooth (likewise $\mathbb{C}$), w/ derivative at 0 given by identity. So exp is a diffeo taking a nbhd of 0 to a nbhd of I, which if small enough lives in $GL(n,\mathbb{R})$



On one side we have a vector space $\rightsquigarrow$ $T_0(M_n(\mathbb{R})) = M_n(\mathbb{R})$

On one side we have the Lie algebra $T_I\, GL(n,\mathbb{R})$

<u>Propn</u> For each of the classical matrix Lie groups, $\exists$ $\mathfrak{g}$ and u, V small enough that



is a diffeomorphism.

$\Rightarrow \mathfrak{g} = T_0 g$ is the Lie algebra.

<u>Examples</u> $SL(n,\mathbb{R})$

- $\det(e^A) = e^{\operatorname{tr} A}$. If $\operatorname{tr} A = 0$, $\det e^A = 1$. And if $\det(e^A) = 1$, $\operatorname{tr}(A) = 2\pi i m$. Near 0, $m = 0$. So near the identity, $SL(n,\mathbb{R})$ is the image of the traceless matrices.

- $SO(n,\mathbb{R})$ If $AA^T = I$, write $A = I + tB + t^2(\ldots)$. Then

$$AA^t = (I + tB + (\ldots))(I + tB^t + (\ldots))$$
$$= I + t(B + B^t) + t^2(\ldots)$$

$\Rightarrow B + B^t = 0$

$\Rightarrow B$ is antisymmetric.

<u>Remark</u> This is a second proof of the dimension of $SO(n,\mathbb{R})$ and indeed a second proof that it is a Lie group.

So exp carries traceless antisymmetric matrices near 0 to orthogonal matrices near I.

<u>Remark</u> For any $A \in M_n(\mathbb{R})$, we get a <u>one-parameter subgroup</u>

$\mathbb{R} \to GL(n,\mathbb{R})$

$t \mapsto \exp(tA)$

In general a one-parameter subgroup of G is a smooth map $\alpha : \mathbb{R} \to G$. $\alpha(\mathbb{R})$ may or may not be a Lie subgroup.

Exercise ① If G is a connected Lie group and U is a neighborhood of 1, U generates G.

② A Lie group homomorphism $F: G_1 \to G_2$ where G_2 is connected which induces a surjection $F_*: \mathfrak{g}_1 \to \mathfrak{g}_2$ is necessarly surjective.

Questions

① How can we generalize the exponential map?

② what is the general relationship between the category of Lie algebras & Lie groups?

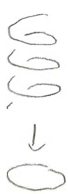
i.e. $\mathfrak{g} \Rightarrow G$?

$F: \mathfrak{g} \to \mathfrak{h} \Rightarrow F: G \to H$?

Observation

① We want to include the word "connected"; $SO(n)$ and $O(n)$ have the same Lie algebra.

② Covering spaces have the same Lie algebras as the groups they cover

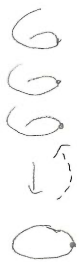


$$\begin{array}{c} SU(2) \\ \downarrow \\ SO(3) \end{array}$$

Remark It's not even true that connected Lie groups w/ same Lie algebra & same fundamental group are the same

$$SO(4) \ncong SO(3) \times SU(2)$$

Remark



$$\begin{array}{ccc} \mathbb{R} & & \mathbb{R} \\ \parallel & & \parallel \\ \mathfrak{g} & \longrightarrow & \mathfrak{g} \\ a & \longmapsto & a \end{array}$$

This doesn't lift to a morphism in the other direction.