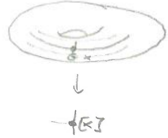
Recall from last time

Thm If G a lie group acts on M smoothly, freely, and properly, M/G is a manifold st $\pi : M \to M/G$ is smooth.



Corollary 1 H a closed immersed Lie subgroup $\Rightarrow$ G/H is a manifold w/ a smooth transitive action of G.

Corollary 2 A closed immersed Lie subgroup $H \leq G$ is embedded in G.

Corollary 3 If G acts on M smoothly & properly, the orbits Gx are regular submfds diffeomorphic to G/G_x $\hookleftarrow$ stabilizer.

eg

$S^{n-1} \cong O(n)/O(n-1)$ $\qquad$ $\mathbb{RP}^{n-1} \cong O(n)/O(1)\times O(n-1)$

$S^{2n-1} \cong U(n)/U(n-1)$ $\qquad$ $\mathbb{CP}^{n-1} \cong U(n)/U(1)\times U(n-1)$

$S^{4n-1} \cong Sp(n)/Sp(n-1)$ $\qquad$ $\mathbb{HP}^{n-1} \cong Sp(n)/Sp(k)\times Sp(n-k)$

Example For $0<k<n \in \mathbb{N}$, $Gr_k \mathbb{R}^n = \{\text{linear subspaces of dim } k \text{ in } \mathbb{R}^n\}$

Any orthonormal basis for $U \leq \mathbb{R}^n$ can be extended to an ONB for $\mathbb{R}^n$ $\Rightarrow$ $O(n)$ acts transitively on $Gr_k \mathbb{R}^n$.

If $U = \text{Span}\{e_1, \dots, e_k\}$, $G_U = \left\{ \begin{array}{cc} & \overbrace{}^{k} \\ k\{ & \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right) \end{array} \in O(n) \right\} = O(k)\times O(n-k)$

$$\Rightarrow Gr_k(\mathbb{R}^n) \simeq O(n) / O(k) \times O(n-k)$$

$$Gr_k \mathbb{C}^n \simeq U(n) / U(k) \times U(n-k)$$

$$Gr_k \mathbb{H}^n \simeq Sp(n) / Sp(k) \times Sp(n-k)$$

The Low-Dimensional Coincidences

Recall we have

$$-1 \simeq \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$1 \longrightarrow SO(n) \longrightarrow O(n) \longrightarrow O(1) \longrightarrow 1$$

$O(1) \simeq \mathbb{Z}/2\mathbb{Z}$

Splits but not a direct product (eg $O(n)$ not abelian)

$O(n) \simeq SO(n) \rtimes \mathbb{Z}/2\mathbb{Z}$

$$1 \longrightarrow SU(n) \longrightarrow U(n) \longrightarrow U(1) \longrightarrow 1$$

$U(1) \simeq S^1$

(No comparable special group for the quaternions; the determinant is not a well-defined homomorphism.)

Complex numbers $a+bi \longleftrightarrow$ Real matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

Quaternions $a+bi+cj+dj = (a+bi)+(c+id)j \longleftrightarrow$ Complex matrices $\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$

Unit quaternions $Sp(1) \hookrightarrow SU(2)$ as $\left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1 \right\}$

For $v = a+bi+cj+dk$, let $Re(v) = a$, $Im(v) = bi+cj+dk$.

Let $\langle , \rangle$ be the normal dot product on $\mathbb{R}^4$, so that

$$\langle \underbrace{a_1+b_1i+c_1j+d_1j}_{v},\ \underbrace{a_2+b_2i+c_2j+d_2k}_{v'} \rangle = a_1a_2+b_1b_2+c_1c_2+d_1d_2 = Re(v\bar{v'}).$$

Note that $Re(\mathbb{H})^{\perp} = Im(\mathbb{H})$.

Now $v \in Sp(1) \curvearrowright \mathbb{H}$ by $v \cdot w = vw\bar{v}$. Note $Re(v1\bar{v}) = 1$ $\quad Re(vj\bar{v}) = 0$

$Re(vi\bar{v}) = 0$ $\quad Re(vk\bar{v}) = 0$

$\uparrow$

$(a+bi+cj+dk)(ai+bi-ck+dj)$

So $Re(v\cdot w) = Re(w)$. But also

$$\langle v\cdot w, v\cdot w' \rangle = Re(vw\bar{v}\,(\overline{vw'\bar{v}}))$$

$$= Re(vw\bar{v}\,\bar{\bar{v}}\,\bar{w}'\,\bar{v})$$

$$= Re(vw\bar{v}v\bar{w}'\bar{v})$$

$$= Re(vw\bar{w}'\bar{v})$$

$$= Re(w\bar{w}')$$

$$= \langle w, w' \rangle$$

So $Sp(1)$ acts orthogonally on $\mathbb{H}$. Hence since it acts orthogonally on $Re(\mathbb{H})$, it also does on $Im(\mathbb{H})$.

This gives a homomorphism $\varphi : Sp(1) \to O(3)$. Since $Sp(1)$ is connected, actually $\varphi : Sp(1) \to SO(3)$.

$$\begin{array}{c} Sp(1) \\ \parallel \\ SU(2) \\ \cong \\ S^3 \end{array}$$

What is $\ker \varphi$? If $v = a+bi+cj+dk$ w/ $a^2+b^2+c^2+d^2=1$, $v \in \ker\varphi$ if $vw\bar{v} = w$ $\forall\, w \in \mathbb{H}$. In particular $vi\bar{v} = i \Rightarrow vi = iv$

$\Rightarrow ai - b - ck + dj = ai - b + ck - dj \Rightarrow c, d = 0$. Likewise $vj = jv \Rightarrow b = 0$,

So $v = a = \pm 1$, and $\ker\varphi = \{\pm 1\}$.

So
$$\begin{tikzcd} Sp(1) \arrow[rr, "\rho"] \arrow[dr] & & SO(3) \\ & Sp(1)/\mathbb{Z}/2\mathbb{Z} \arrow[ur] & \end{tikzcd}$$
, As $\mathbb{Z}/2\mathbb{Z}$ is normal, $Sp(1)/\mathbb{Z}/2\mathbb{Z}$ is a

Lie group, so $\rho : Sp(1) \longrightarrow SO(3) \simeq Sp(1)/(\mathbb{Z}/2\mathbb{Z})$ is smooth & has

constant rank & is bijective, hence is an immersion.

So $SO(3) \simeq SU(2)/(\mathbb{Z}/2\mathbb{Z})$ as Lie groups. $\begin{matrix} SU(2) \simeq S^3 \\ \downarrow \\ SO(3) \simeq \mathbb{R}P^3 \end{matrix}$, This is a

double cover. $\Big(\pi_1(S^3) = 0$ and $\pi_1(SO(3))$ is generated by $\gamma(t) = \begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) & 0 \\ -\sin(2\pi t) & \cos(2\pi t) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The Hopf Fibration

Now $\overset{G=}{SO(3)} \circlearrowright S^3$ transitively w/ $G_{e_1} = \{ g \in SO(3) : g e_1 = e_1 \} \simeq SO(2) \simeq S^1$

Hence $S^2 \simeq SO(3)/SO(2)$.

It follows that $SU(2)$ acts on S^2 transitively, and

$$\rho^{-1}(SO(2)) = \{ v \in Sp(1) : v i \bar{v} = i \} = \left\{ \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix} : a+bi \in U(1) \right\}$$

$$\Rightarrow S^2 \simeq SU(2)/U(1) \simeq S^3/S^1$$

So there is a smooth map $F: S^3 \longrightarrow S^2$ w/ kernel S^1.
$A \longmapsto Ae_1$

This is the Hopf fibration. (Note that this is not a product, any invariant of algebraic topology shows $S^2 \times S^1 \not\simeq S^3$.)

Similarly, the structure of $SO(4)$

Dimension $\frac{4(4-1)}{2} = 6$.

Let $Sp(1) \times Sp(1) \curvearrowright \mathbb{H}$ by $(u,v)\cdot w = uw\bar{v}$. Then

$$\begin{aligned}
\langle (u,v)\cdot w, (u,v)\cdot w' \rangle &= Re(uw\bar{v}\,\overline{(uw'\bar{v})}) \\
&= Re(uw\bar{v}v\bar{w}'\bar{u}) \\
&= Re(uw\bar{w}'\bar{u}) \\
&= Re(w\bar{w}') \\
&= \langle w, w' \rangle
\end{aligned}$$

As before $(u,v) \longmapsto (u,v)\cdot$ defines a homomorphism $\psi: Sp(1)\times Sp(1) \longrightarrow SO(4)$. What is $\ker \psi$? $uw\bar{v} = w \;\forall\, w \in \mathbb{H}$

$\Rightarrow uw = wv \quad \forall\, w \in \mathbb{H}$ including 1

$\Rightarrow u = v \;\Rightarrow u = \pm 1$ by last time

So $\ker \psi = \mathbb{Z}/2\mathbb{Z}$

Ergo $\frac{Sp(1)\times Sp(1)}{\mathbb{Z}/2\mathbb{Z}} \longrightarrow SO(4)$ is an embedding, and both are connected and of dimn 6, so $SO(4) \simeq Sp(1)\times Sp(1) / \{(I,I),(-I,-I)\}$

Application to Grassmanians

Know $SO(4) \circlearrowright Gr_2\mathbb{R}^4$ transitively

$\Rightarrow SU(2)\times SU(2) \circlearrowright Gr_2\mathbb{R}^4 \simeq \{\text{real 2-d subspaces of } \mathbb{H}\}$

$(SU(2)\times SU(2))_{\langle 1,i\rangle} = \underbrace{U(1)\times U(1)}_{\text{the diagonal matrices}}$ where $U(1) \simeq \left\{\begin{pmatrix} q & 0 \\ 0 & \bar{q}\end{pmatrix}\right\}$ correspond to $a+bi+0j+0k$

Hence $Gr_2\mathbb{R}^4 \simeq SU(2)\times SU(2)/U(1)\times U(1) \simeq \overset{S^3}{\overset{\shortparallel}{SU(2)}}/\underset{S^1}{\underset{\shortparallel}{U(1)}} \times SU(2)/U(1) \simeq S^2\times S^2$

One more $Sp(2)/\mathbb{Z}/2\mathbb{Z} \simeq SO(5)$ (Harder.)

Vector Fields $\rightsquigarrow$ Lie Algebras

__Defn__ A __vector field__ on M is a function $Y : M \to \coprod_{p\in M} T_p M$

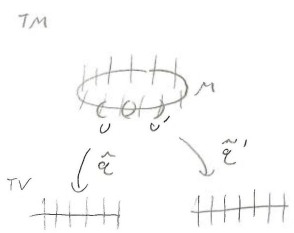
st (1) $\forall\ p \in M,\ Y(p) \in T_p M$

(2) $\forall$ chart $\varphi : U \to V \subseteq \mathbb{R}^n$, if $\theta : \coprod_{v\in V} T_v \mathbb{R}^n \to \mathbb{R}^n$ is the natural map,

$\theta \circ \varphi_* \circ Y|_U : U \to \mathbb{R}^n$ is smooth.

In particular, if $M \subseteq \mathbb{R}^n$ is open, $Y(x) = \sum_{i=1}^{n} y_i(x) \frac{\partial}{\partial x_i}$ for $y_i \in C^\infty(M)$.

__Defn__ The __tangent bundle__ to M is $TM := \bigsqcup_{p\in M} T_p M$ equipped w/ the structure of a $2n$-dim'l mfd as follows: For each chart $\varphi : U \to V \subseteq \mathbb{R}^n$ in an atlas for M, let $\hat{U} = \bigsqcup_{p\in U} T_p M$, and $\hat{V} = \bigsqcup_{q\in V} T_q \mathbb{R}^n \simeq V \times \mathbb{R}^n$ an open subset of $\mathbb{R}^{2n}$. Then take $\tilde{\varphi} : \hat{U} \to \hat{V}$ to be $(\varphi(p), D_p\varphi(u))$ for $u \in T_p M$.



φ a diffeomorphism $\Rightarrow$ $\tilde{\varphi}$ is a bijection.

And if $\varphi : U \to V$, $\varphi' : U' \to V'$, $q = \varphi(p) \in \varphi(U')$,

$$\tilde{\varphi}' \circ \tilde{\varphi}(p) = \tilde{\varphi}'(D_p\varphi)^{-1} v$$

$$= (\varphi'(p), D_p\varphi' \circ D_p\varphi^{-1} v)$$

$$= (\varphi' \circ \varphi^{-1}(q), D_q(\varphi' \circ \varphi^{-1}) v)$$

both smooth by construction

So $\tilde{\varphi}' \circ \hat{\varphi}$ is smooth where it exists.

__Example__ $TS^2 \not\cong S^2 \times \mathbb{R}^2$

Clearly $\pi : TM \to M$ is smooth.

Exercise $F : M \to N$ smooth induces smooth $f_* = DF : TM \to TN$.

Defn A vector field on M is a smooth $Y : M \to TM$ st $\pi \circ Y = id_M$.

We can add two vector fields in a sensible way; the set of vector fields on M is a vector space $VF(M)$.

Global Derivations

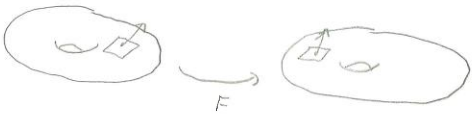
Defn A global derivation on M is a linear map $Y : C^\infty(M) \to C^\infty(M)$ satisfying $Y(fg) = Y(f)g + fY(g)$.

$GD(M)$ is also a vector space.

Thm $GD(M) \simeq VF(M)$

Pf Smooth vector field $Y = \sum_{i=1}^n y_i(x) \frac{\partial}{\partial x_i}$ in each chart U, so for $f \in C^\infty(M)$, $Yf|_U = \sum_{i=1}^n y_i(\vec{x}) \frac{\partial f}{\partial x_i}(\vec{x}) \in C^\infty(M) \Rightarrow Yf \in C^\infty(M)$. This gives a natural map $VF(M) \to GD(M)$. Now if $Y \in GD(M)$, we claim that the map $p \mapsto Y(\;)(p)$ is a smooth vector field. Locally $(Y \cdot)\vec{x}^{-1}$ is $\sum_{i=1}^n y_i(x) \frac{\partial}{\partial x_i}$ for some functions y_i. Suffices to check these fcns are smooth. Let x_j be coordinate fcns on U, and extend to M using a bump fcn (see hw), so $Yx_j \in C^\infty(M)$. Then $(Y x_j)(\vec{x}) = \sum_{i=1}^n y_i(\vec{x}) \frac{\partial}{\partial x_i}(x_j) = y_j(\vec{x}) \in C^\infty(M)$.

Def^{n} If $Y \in VF(M)$, $F: M \to N$ a diffeomorphism, the push-forward $FY \in VF(N)$ is the vector field such that $FY(F(p)) = F_*(Y(p))$.



(*) Lemma from page 10 here

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Smoothness $FY = F_* \circ Y \circ F^{-1}$ is smooth.

The Lie bracket

If $X, Y \in VF(M) = GD(M)$, then $f \mapsto XY(f)$ is not in general in $GD(M)$.

$$\begin{aligned} XY(fg) &= X((Yf)g + f(Yg)) \\ &= X(Yf)g + Xf(Yg) \\ &= (XYf)g + \underline{(Yf)(Xg)} + \underline{(Xf)(Yg)} + f(XYg) \end{aligned}$$

But $f \mapsto XYf - YXf \in GD(M)$ does satisfy the Liebniz rule. (Notice the unfortunate terms above are symmetric)

So X, Y define a new vector field called the Lie bracket.

Case of $\mathbb{R}^n$

Open $U \subseteq \mathbb{R}^n$, $X = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$, $Y = \sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}$

$$\begin{aligned} XYh - YXh &= \sum_{i,j} f_i \frac{\partial}{\partial x_i}\left(g_j \frac{\partial}{\partial x_j} h\right) - g_j \frac{\partial}{\partial x_j}\left(f_i \frac{\partial h}{\partial x_i}\right) \\ &= \left(\sum_{i,j} f_i \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j} \frac{\partial}{\partial x_i}\right) h \end{aligned}$$

<u>Propn</u> $F: M \to N$ a diffeomorphisms $\Rightarrow F[X,Y] = [FX, FY]$

<u>Proof</u> For $g \in C^\infty(N)$, $[FX, FY]g = (FX)(FY)g - (FY)(FX)g$

$$= FX\,(Y(g\circ F)\circ F^{-1}) - FY\,(X(g\circ F)\circ F^{-1})$$

$$= X\,(Y(g\circ F)\circ F^{-1}\circ F)\circ F^{-1} - Y\,(X\circ(g\circ F)\circ F^{-1}\circ F)\circ F^{-1}$$

$$= XY\,(g\circ F)\circ F^{-1} - YX\,(g\circ F)\circ F^{-1}$$

$$= [X,Y]\,(g\circ F)\circ F^{-1}$$

$$= F[X,Y]g$$

(*)

<u>Lemma</u> As a GD, $FY(g) = Y(g\circ F)\circ F^{-1}$!

<u>PF</u> $(FY)(g)(F(p)) = FY(F(p))g$

$$= F_* Y(p)g$$

$$= Y(p)(g\circ F)$$

$$= Y(g\circ F)(p)$$

$$= Y(g\circ F)\,(F^{-1}(F(p)))$$

<u>DeFn</u> A <u>Lie algebra</u> is a real vector space equipped w/ a bilinear operation $X, Y \mapsto [X,Y]$ satisfying

1. Antisymmetry $[X,Y] = -[Y,X]$
2. Jacobi identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

Lie algebras are not algebras.

Example 0 Any vector space w/ $[v_1, v_2] = 0$.

Example 1 Any algebra w/ $[x,y] = xy - yx$.

$[[x,y],z] + [[y,z]x] + [[z,x],y] = 0$ after writing out twelve cancelling terms

Example 2 $VF(M)$ on any smooth M. The Jacobi identity follows from the fact that XYF is a sensible term even if XY isn't.

Example 3 Any subspace of a Lie algebra closed under the Lie bracket.

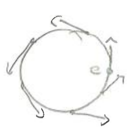
$\{A \in M_n(\mathbb{R}) : tr(A) = 0\}$ $\quad tr(AB - BA) = tr(AB) - tr(BA) = 0$.

Example 4 G a Lie group, $L_g : G \to G$, $h \mapsto gh$. Say $X \in VF(G)$ is left-invariant if $\forall\, g \in G$, $L_g X = X$. Let $Lie(G) := \left\{ \begin{array}{l} \text{Left-invt vector} \\ \text{fields on } G \end{array} \right\}$.

Propn $Lie(G)$ is closed under $[\,,\,]$

PF $L_g[X,Y] = [L_g X, L_g Y]$
$= [X,Y]$

Example $G = S^1$



Left invt vector field has same magnitude & orientation at each point.