Group actions on manifolds

Goal Show M/G is a smooth mfd in good cases.

$\rightsquigarrow$ Show the quotient of a Lie group by a closed Lie subgroup is a manifold, and use this to get more examples

Defn A (smooth) action of a Lie group G on a manifold M is a left action $G \times M \longrightarrow M$ which is also smooth as a map of manifolds.

Note $\forall\, g \in G$, $m \mapsto g \cdot m$ is a diffeomorphism.

Examples

① $GL(n, \mathbb{R}) \circlearrowright \mathbb{R}^n$

② $O(n) \circlearrowright \mathbb{R}^n$

③ $O(n) \circlearrowright S^{n-1}$

④ G acts on H normal by gHg^{-1}] the adjoint action

Rank Thm If $f: M \longrightarrow N$ has $\operatorname{rank}(D_x f) = k$ for all x in some nbhd of a point $p \in M$, then $\exists$ charts $\varphi: U \longrightarrow V$ on M and $\psi: U' \longrightarrow V'$ on N st $\varphi(p) = 0 = \psi(f(p))$ so that $\psi \circ f \circ \varphi^{-1}((x_1, \dots, x_m)) = (x_1, \dots, x_k, 0, \dots, 0)$.

Pf Wlog $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, $p = 0$, $f(p) = 0$. Permute the coordinates so the upper left minor of $D_x f$ is nonsingular. Let $\vec{u} \in \mathbb{R}^k$, $\vec{v} \in \mathbb{R}^{m-k}$, $f(\vec{u}, \vec{v}) = (\underbrace{g(\vec{u}, \vec{v})}_{k}, \underbrace{h(\vec{u}, \vec{v})}_{n-k})$ w/ $D_0 g = \left(\begin{array}{c|c} k \times k & * \end{array}\right)$ where the $k \times k$ block is nonsingular.

Define $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ by $\varphi(\vec u, \vec v) = (g(\vec u, \vec v), \vec v)$. $D_0 \varphi = \left(\begin{array}{c|c} \frac{dg}{du} & * \\ \hline 0 & I \end{array}\right)$

We apply the inverse fcn thm to conclude that $\exists$ local $\varphi^{-1}(\vec u, \vec v) = (q(\vec u, \vec v), \vec v)$ defined on a nbhd of 0 in $\mathbb{R}^m$.

Then $D_x f$ has rank k for x near 0 $\Rightarrow$ $D_y(F \circ \varphi^{-1})$ is also of rank k for y near 0. But $D_y(F \circ \varphi^{-1})(\vec u, \vec v) = (g \circ (q(\vec u, \vec v), \vec v), h(q(\vec u, \vec v)) \vec v)$

$$= (\vec u, h(q(\vec u, \vec v), \vec v))$$

$$= \left(\begin{array}{c|c} I & 0 \\ \hline ? & * \end{array}\right)$$

$(*)$ must vanish since the rank of this matrix is k. So on a small ball around 0, $h(q(\vec u, \vec v), \vec v)$ is independent of $\vec v \Rightarrow h(q(\vec u, \vec v), \vec v) = h(q(\vec u, 0), 0) := r(\vec u)$

So $F \circ \varphi^{-1}(\vec u, \vec v) = (\vec u, r(\vec u))$. Now let $\psi(\vec u, \vec w) = (\vec u, \vec w - r(\vec u))$. Then $D_0 \psi = \left(\begin{array}{c|c} I & 0 \\ \hline * & I \end{array}\right)$

ψ is a diffeomorphism by the inverse fcn thm. And $\psi \circ F \circ \varphi^{-1}(\vec u, \vec v) = (\vec u, 0)$ as desired.

Corollary $F : M \to N$ smooth & injective w/ $D_x F$ of constant rank $\Rightarrow$ F is an immersion.

Pf If not, locally of the form $(x_1, \dots, x_m) \mapsto (x_1, \dots, x_k, 0, \dots, 0)$, which is not injective on any nbhd of M.

Proper Maps

Recall For X, Y topological spaces, a cts map $F : X \to Y$ is proper if the inverse images of compact sets are compact.

Note ① Closed $X \subseteq Y$, $X \hookrightarrow Y$ is proper

② X cpt, Y Hausdorff $\Rightarrow$ any $f: X \to Y$ is proper.

③ Homeomorphisms are proper

④ Compositions of proper maps are proper.

Exercises X, Y are topological mfds and $f: X \to Y$ proper $\Rightarrow f$ is a closed map.

Corollary A proper injective immersion is an embedding.

Defn A group action is proper if $\mu: G \times M \to M \times M$, $(g, x) \mapsto (g \cdot x, x)$ is a proper map of topological spaces,

Examples ① The left action of G on itself is a proper map.

$G \times G \to G \times G$, $(g, h) \mapsto (gh, h)$ is a diffeomorphism w/ inverse $G \times G \to G \times G$, $(g, h) \mapsto (g^{-1}h, h)$

② A closed subgroup H acts properly on G; $H \times G$ is closed in $G \times G$.

③ $O(n)$ acts properly on $\mathbb{R}^n$: If $C \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is cpt, $\underset{\text{closed}}{\mu^{-1}(C)} \subseteq \underset{\text{cpt}}{O(n)} \times \underset{\text{cpt}}{\pi_2(C)}$

$\Rightarrow \mu^{-1}(C)$ is cpt

Likewise for any action of a compact group G.

Not Example $GL(2, \mathbb{R})$ on $\mathbb{R}^2$. $G = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$ is a closed subgroup

but the inverse image of $\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ under $G \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$, $(g, x) \mapsto (gx, x)$ is $G \times \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ is non cpt.

So the action of $GL(2, \mathbb{R})$ isn't proper either.

Thm! If a Lie group acts smoothly, freely, and properly on a manifold M, M/G is smooth and $\pi : M \to M/G$ is a smooth map. Moreover $\dim(M/G) = \dim M - \dim G$.

Pf The quotient map is open, so if $\exists$ a countable basis for the topology upstairs, its projection is a countable basis for the topology downstairs. So M/G is second countable. For Hausdorff, the image of $\mu : G \times M \to M \times M$ is $S = \{(x,y) \in M \times M : y = gx \text{ for some } g \in G\}$, which is closed b/c proper maps are closed. If $[x] = [y]$ in M/G, $(x,y) \notin S$, so $\exists$ $U \times V \subseteq M \times M \setminus S$ open containing (x,y). So $[x] \in \pi(U)$, $[y] \in \pi(V)$, $\pi(U) \pitchfork \pi(V) = \emptyset$.

To produce a chart near $[x] \in M/G$, we go through a few steps.

Claim 1 $i : G \to M$, $x \mapsto gx$ is an embedding.

$$\begin{tikzcd} G \arrow[r, "h\cdot"] \arrow[d, "i"'] & G \arrow[d, "i"] \\ M \arrow[r, "h\cdot"] & M \end{tikzcd} \qquad \begin{tikzcd} T_eG \arrow[r, "D_e(h\cdot)", "\sim"'] \arrow[d, "D_ei"'] & T_hG \arrow[d, "D_hi"] \\ T_xM \arrow[r, "\sim", "T_x(h\cdot)"'] & T_{h\cdot x}M \end{tikzcd}$$

Horizontal maps are isomorphisms, $rk(D_e i) = rk(D_h i) \Rightarrow rk(D_h i)$ is independent of h. And G acts freely, so i is injective. Hence by the rank thm i is an immersion.

But $\mu : G \times M \to M \times M$ is proper, and $i = \pi_1 \circ \mu|_{G \times \{x\}}$. But we see

$$\begin{tikzcd} F := \mu|_{G\times\{x\}} : G \times \{x\} \arrow[r] & M \times \{x\} \arrow[d, "\pi_1"] \\ & M \end{tikzcd}$$

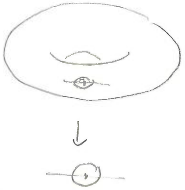
is proper, so i is proper. Hence i is an embedding

Claim 2 $\forall$ $x \in M$, $\exists$ an embedding $G \times B_1(0) \hookrightarrow M$, $(g, 0) \mapsto x$ st

$$\begin{tikzcd} G \times B_1(0) \arrow[r, hook] \arrow[d, "(g\cdot, id)"'] & M \arrow[d, "g\cdot"] \\ (G \times B_1(0)) \arrow[r, hook] & M \end{tikzcd}$$

PF i is an embedding $\Rightarrow$ $i(G) = Gx$ is a regular submanifold.

Let $\underset{M}{\overset{\cap\mathrm{I}}{\varrho : U}} \longrightarrow \underset{\mathbb{R}^m}{\overset{\cap\mathrm{I}}{V}}$; $\varrho(x) = 0$, $\varrho(Gx) = (\mathbb{R}^k \times 0) \cap V$, $k = \dim G$. Choose $\varepsilon > 0$ st

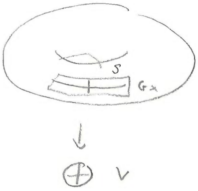


$0 \times B_\varepsilon(0) \subseteq V$, and let $S = \varrho^{-1}(0 \times \varrho_\varepsilon(0))$, $\bar{S} = \varrho^{-1}(0 \times \overline{B_\varepsilon(0)})$.

Note that $\bar{S} \cap Gx = \{x\}$.

(Mild abuse of notation: $S = B_\varepsilon(0)$, so that $0 \in S$ refers to x, and $\frac{1}{2}S = B_{\frac{\varepsilon}{2}}(0)$.)

Then
$$\begin{tikzcd} G \times \bar{S} \arrow[r] \arrow[rr, bend right, "F"'] & M \times \bar{S} \arrow[r] & M \end{tikzcd}$$
is a proper map, so



$F^{-1}(\bar{S} \setminus S)$ is compact, and disjoint from $G \times \{0\}$

$\Rightarrow \pi_1(F^{-1}(\bar{S} \setminus S))$ is cpt & disjoint from 0. Choose a smaller ball $T \subseteq S$ centered at 0 not meeting $\pi_1 F^{-1}(\bar{S} \setminus S)$. Replace F w/ $F|_{G \times T}$ so that $F^{-1}(\bar{S}) = F^{-1}(S)$. By construction $D_{e,x}F$ is an isomorphism $\Rightarrow$ by IFT, after possibly shrinking T, $D_{e,t}F$ is an isomorphism for all $t \in T$. By the usual argument

$$\begin{tikzcd} T_eG \times T_0\bar{S} \arrow[r, "D_e g\cdot \oplus \mathrm{id}"'] \arrow[d, "D_{e,t}F"'] & T_gG \times T_0(\bar{S}) \arrow[d, "D_{g,e}F"] \\ T_xM \arrow[r, "D_x g\cdot"'] & T_{g\cdot x}M \end{tikzcd}$$

$\Rightarrow \forall\, g \in G$, $t \in T$, $D_{g,t}F$ is an isomorphism. So F is an open immersion $F : G \times T \to M$. We claim we can shrink T st F is injective, hence an embedding.

$\Rightarrow F$ is an open immersion, hence a submersion, so $F^{-1}(S)$ is a regular submfd of dimn $m-k$. So is $\{e\} \times T$. Replace T by $\frac{1}{2}\bar{T}$ and F by $F|_{G \times \frac{1}{2}\bar{T}}$.

$G \times \frac{1}{2}\bar{T}$ is closed in $G \times M$ $\Rightarrow$ $F^{-1}(S) \subseteq G \times \frac{1}{2}\bar{T}$ is compact, and $\{e\} \times \frac{1}{2}T$ open in $F^{-1}(S) \Rightarrow F^{-1}(S) \setminus (\{e\} \times \frac{1}{2}\bar{T})$ is cpt in $G \times \frac{1}{2}\bar{T}$ and disjoint from $G \times 0$. So $\pi_2(F^{-1}(S) \setminus \{e\} \times \frac{1}{2}\bar{T})$ is cpt in $\frac{1}{2}\bar{T}$ and does not meet $0 \Rightarrow$ we can shrink to a smaller ball U st $F|_{G \times U}$ is injective. And an injective open immersion is an embedding.

__Finish up__ Define a chart $\varphi : \pi(V) \longrightarrow U$, This is a well-defined

$$Gy \longmapsto \pi_2(F^{-1}(y))$$

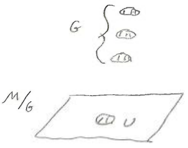
local homeomorphism, overlap maps between V' and V are

$$\begin{tikzcd}
V' \arrow[r, hook] & M \\
V' \cap V \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, hook] & V \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, "F^{-1}"] & G \times U \arrow[r, "\pi_2"] & U
\end{tikzcd}$$

hence smooth.

__Corollary 0__ A free smooth quotient of a manifold by a finite group is a manifold, and the quotient map is a covering map.

__PF__ Finite groups are compact; actions by compact groups are proper. Covering space structure given by the lemma: $V \subseteq M$ diffeomorphic to $G \times U \subseteq G \times M/G$.



__Corollary 1__ If $H \leq G$ is a closed immersed Lie subgroup, the space of left cosets G/H is a manifold on which G acts smoothly and transitively. These are __homogeneous spaces__.

__PF__ $\begin{aligned} G \times G &\longrightarrow G \times G \\ h \times g &\mapsto gh^{-1} \times g \end{aligned}$ is a diffeomorphism, hence proper. So is the restriction

$H\times G \to G\times G$ if H is closed, so G/H is a manifold. Obvious left action is transitive and given by

$$\begin{tikzcd}
 & V \arrow[d, phantom, "\rotatebox{-90}{$\subseteq$}"] \arrow[dr, "F^{-1}"] & & \\
G\times U \subseteq G\times G \arrow[r, "m"] & G & H\times U' \arrow[dr, "\pi_2"] & \\
(g, hG) \mapsto gh \longmapsto ghG & & & U'
\end{tikzcd}$$

Corollary 2 A closed immersed Lie subgroup of a Lie group G is embedded in G.

PF $\forall\ h\in H$, choose a slice S st $h\in S\subseteq G$ and $i_* T_h H \oplus T_h S = T_h G$. Corollary 1 tells us $G \to G/H$ is a submersion, but $H \xrightarrow[i]{} G \xrightarrow[\pi]{} G/H$ is constant, so $\pi_* i_* = 0$

$$\Rightarrow i_* T_h H \subset \ker \pi_*$$

$$\Rightarrow \pi_*\big|_{T_h S} : T_h S \xrightarrow{\sim} T_{eH}\, G/H$$

$$\Rightarrow \pi|_S : S \to G/H \text{ is a local diffeo near } h.$$

Shrink S st $S\cap H = \{h\}$. Choose a Euclidean nbhd $U\subseteq H$ of the identity $\{e\}$, then $\begin{matrix} F: U\times S \to G \\ (g, g') \mapsto gg' \end{matrix}$ is a diffeo near H st $F^{-1}(H) = U\times H$.

<u>Corollary 3</u> If G acts on M smoothly and properly (but perhaps not freely) the orbits $Gx \subseteq M$ are regular mfds diffeomorphic to G/G_x, where $G_x = \{g \in G : gx = x\}$ is the stabilizer or isotropy group of $x \in M$.

<u>PF</u> Let $H = G_x$. The map $\mu : G \times M \to M \times M$ is proper $\Rightarrow \mu^{-1}(\{x\}, \{x\}) = G_x \times \{x\} = H$ is compact $\Rightarrow H$ is closed. Then $\begin{matrix} i : G/H \to M \\ gH \mapsto g \cdot x \end{matrix}$ is well-defined and injective w/ image Gx. Charts on G/H are given by slices $U \subseteq G$, $\Rightarrow f$ smooth.

$$\begin{tikzcd}
U \arrow[r, phantom, "\subseteq"] & G \arrow[dr, bend left] & g \arrow[d, mapsto] \\
G/H \arrow[u] \arrow[rr] & & M \quad g \cdot x
\end{tikzcd}$$

By the usual argument, rank $(D_{gH}\, i)$ is independent of g. By the corollary to the rank thm, $i : G/H \to M$ is an immersion.

Then $C \subseteq M$ compact $\Rightarrow \mu : G \times M \to M \times M$ proper, so $\mu^{-1}(C \times \{x\})$ is cpt

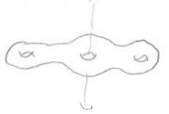
But
$$\begin{tikzcd}
G \arrow[r, "\pi"] \arrow[dr, "\tilde{i}"] & G/H \arrow[d, "i"] \\
g \mapsto g \cdot x & M
\end{tikzcd}$$
$\tilde{i}^{-1}(C) = \mu^{-1}(C \times \{x\})$ is compact, so $i^{-1}(C) = \pi\, \tilde{i}^{-1}(C)$ is cpt $\rightsquigarrow i$ is proper $\rightsquigarrow$ a proper injective immersion is an embedding.

<u>Counterexample</u> to Corollaries 1-3 for actions which are not proper.

$\mathbb{R} \curvearrowright T^2$ by $x \cdot (e^{t_1}, e^{t_2}) = (e^{ixt_1}, e^{ixt_2})$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Examples

①



$(x,y,z) \longmapsto (-x,-y,-z)$

is a free action by ±1

Quotient is a 2-dim'l mfd double covered by this one.

② $O(n) \curvearrowright \mathbb{R}^n$

$x = 0 \rightsquigarrow G_x = O(n)$

$x = e_1 \rightsquigarrow G_x = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix} \qquad * \in O(n-1)$

$G \cdot x = \{ y \in \mathbb{R}^n : \|y\| = 1 \} = S^{n-1} \quad \Rightarrow \quad S^{n-1} \simeq O(n)/O(n-1)$

③ $U(n) \curvearrowright \mathbb{C}^n \simeq \mathbb{R}^{2n}$

$x = 0 \rightsquigarrow G_x = U(n)$

$S^{2n-1} \simeq U(n)/U(n-1)$

$x = e_1 \rightsquigarrow G_x = U(n-1)$

$G \cdot x = \{ y \in \mathbb{C}^n : \|y\| = 1 \}$

③ Likewise $S^{4n-1} \simeq Sp(n)/Sp(n-1)$

④ $X = \mathbb{R}\mathbb{P}^{n-1} := \{\text{lines through } 0 \in \mathbb{R}^n\}$. Let $G = O(n) \curvearrowright X$ transitively.

$\ell = \langle e_1 \rangle \Rightarrow G_\ell = \left\{ \left(\begin{array}{c|ccc} \pm 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right) \middle| \; * \in O(n-1) \right\} \simeq \mathbb{Z}_2 \times O(n-1) \simeq O(1) \times O(n-1)$

$\Rightarrow \mathbb{R}\mathbb{P}^{n-1} \simeq O(n)/O(1) \times O(n-1)$

Likewise $\mathbb{C}\mathbb{P}^{n-1} := \{\text{lines through } 0 \in \mathbb{C}^n\} \simeq U(n)/U(1) \times U(n-1)$

$\mathbb{H}\mathbb{P}^{n-1} := \{\text{lines through } 0 \in \mathbb{H}^n\} \simeq Sp(n)/Sp(1) \times Sp(n-1)$

<u>Example</u> Let $0 < k < n \in \mathbb{N}$, $X = \{\text{linear subspaces of dimn } k \text{ in } \mathbb{R}^n\}$

$:= \text{Grassmanian } Gr_k \mathbb{R}^n$

<u>eg</u> $\mathbb{R}\mathbb{P}^{n-1} = Gr_1 \mathbb{R}^n$

Any orthonormal basis for $U \subseteq \mathbb{R}^n$ can be extended to an ONB for $\mathbb{R}^n \Rightarrow O(n)$ acts transitively on $Gr_k \mathbb{R}^n$.

If $u = \langle e_1, \ldots, e_k \rangle$ is the span of $e_1, \ldots, e_k$, $G_u = \left\{ k\left\{ \overbrace{\left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array} \right)}^{k} \in O(n) \right\}$

$= O(k) \times O(n-k)$

$\Rightarrow G_k \mathbb{R}^n \simeq O(n)/O(k) \times O(n-k)$

<u>Likewise</u> $Gr_k \mathbb{C}^n \simeq U(n)/U(k) \times U(n-k)$

$Gr_k \mathbb{H}^n \simeq Sp(n)/Sp(k) \times Sp(n-k)$