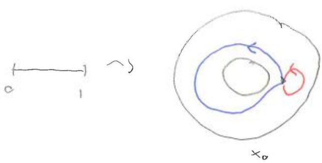
Basic Lie groups ctd: This week, covering spaces and group actions on manifolds.

Recall that the fundamental group of a topological space is the group of paths.

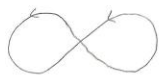
$$\pi_1(X, x_0) = \left\{ \begin{array}{l} \gamma : [0,1] \longrightarrow X \\ \gamma(0) = \gamma(1) = x_0 \end{array} \right\} \Big/ \begin{array}{l} \exists \text{ a cts homotopy} \\ H : [0,1] \times [0,1] \longrightarrow X \\ \quad (s,t) \longmapsto h_s(t) \\ h_0 = \gamma_0,\ h_1 = \gamma_1,\ h_s(0) = x_0 \end{array}$$

w/ multiplication $[\gamma_1]\cdot[\gamma_2] = [\gamma_1 * \gamma_2]$ where $\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$

Examples

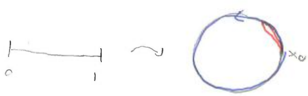


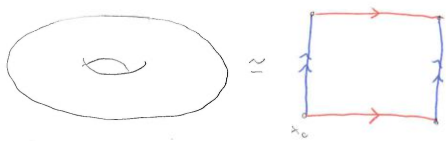
$\pi_1(X, x_0) = \mathbb{Z}$



$\pi_1(X, x_0) = \mathbb{Z} * \mathbb{Z}$] In general nonabelian

Likewise





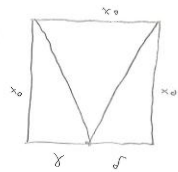
$\pi_1(X, x_0) = \langle \alpha, \beta : \alpha\beta\alpha^{-1}\beta^{-1} = 1 \rangle$

A path-connected space w/ $\pi_1(X, x_0) = 1$ is called simply connected. $\rightarrow$

$\pi_1(X, x_0) = 1$

Example of checking group structure : inverses

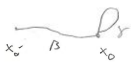
Let $\delta : [0,1] \longrightarrow (X, x_0)$. The homotopy is
$t \longmapsto \gamma(1-t)$



$h_s : [0,1] \to X$

$$t \mapsto \begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2}(1-s) \\ x_0 & \frac{1}{2}(1-s) \le t \le \frac{1}{2}(1+s) \\ \delta(2t-1) & \frac{1}{2}(1+s) \le t \le 1 \end{cases}$$

- A cts map $F : (X, x_0) \longrightarrow (Y, y_0)$ induces a group homomorphism $F_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$
- Changing the basepoint in a path-ctd space conjugates the group

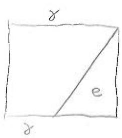


$\pi_1(X, x_0) \longrightarrow \pi_1(X, x_0')$

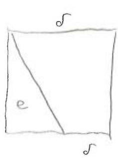
$[\gamma] \longmapsto [\beta * \gamma * \beta^{-1}]$

Propn G a connected Lie group $\Rightarrow \pi_1(G)$ abelian.

Pf γ, δ loops based at $e \in G$. Then $\gamma\delta := \gamma(t)\delta(t)$ is also a loop based at e, and if



$$H(s,t) = \begin{cases} \gamma((2-s)t) & \text{if } (2-s)t \le 1 \\ e & \text{if } (2-s)t \ge 1 \end{cases}$$ and



$$J(s,t) = \begin{cases} \delta((2-s)t - (1-s)) & \text{if } (2-s)t - (1-s) \ge 1 \\ e & \text{if } (2-s)t - (1-s) \le 1 \end{cases}$$

Then $HJ(1,t) = \gamma(t)\delta(t)$

$HJ(0,t) = \gamma * \delta$

$JH(1,t) = \delta(t)\gamma(t)$

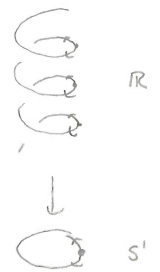
$JH(0,t) = \gamma * \delta$

So $\gamma * \delta \sim \gamma\delta \sim \delta\gamma$. Likewise $\delta * \gamma \sim \gamma\delta \sim \delta\gamma$.

So $[\gamma] \cdot [\delta] = [\delta] \cdot [\gamma] \rightsquigarrow \pi_1(G, e)$ is abelian.

<u>Recall</u> A covering space for X a topological space is a surjective map $p: C \to X$ st $\forall\ x \in X$ $\exists$ an open nbhd U of x st $p^{-1}(U)$ is a union of disjoint open $V_\alpha \in C$ st $p|_{V_\alpha}: V_\alpha \to U$ is a homeomorphism for all α.

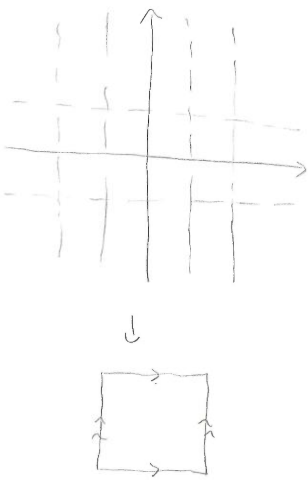
<u>Example</u> 1 Covering spaces of the circle.



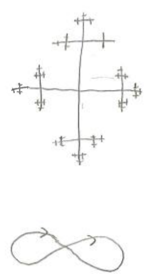
$$P_*: \pi_1(\mathbb{R}) \to \pi_1(S^1)$$
$$0 \mapsto 0$$

$$P_*: \underset{(\mathbb{Z},+)}{\pi_1(S^1)} \to \underset{(\mathbb{Z},+)}{\pi_1(S^1)}$$
$$1 \mapsto n$$

2 $\mathbb{R}^2$ covers the torus



3 A covering space of the wedge of circles



Propn Any covering space of a smooth mfd is also a smooth mfd in such a way that the covering map is smooth.

Defn Two covering spaces of X are isomorphic if there exists a covering space isomorphism between them.

$$\begin{tikzcd} C_1 \arrow[rr, "F"] \arrow[dr, "p_1"'] & \arrow[d, phantom, "\circlearrowleft"] & C_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Thm (From algebraic topology) The covering spaces of a path-connected, locally path-connected, and semilocally simply-connected (every point has a simply-connected neighborhood) space X are in bijection w/ the conjugacy classes of $\pi_1(X, x)$, w/ $\{p : C \longrightarrow X\} \longleftrightarrow \{\text{Conjugacy class of } p_*(\pi_*(C)) \subseteq \pi_*(X)\}$

Universal Cover

$$\tilde{X} = C_e = \{\text{Paths } \gamma : [0,1] \longrightarrow X : \gamma(0) = x_0\}$$

$$\begin{array}{cc} \downarrow & \gamma \\ & \downarrow \\ X & \gamma(1) \end{array}$$

where the topology of $\tilde{X}$ has a basis $U_{[\gamma]} = \{[\gamma * \eta] : \eta \text{ is a path in simply-connected } U \ni \gamma(1) \text{ w/ } \eta(0) = \gamma(1)\}$

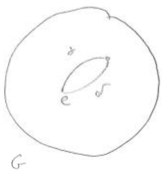
This is simply-ctd and unique up to isomorphism.

$H \subseteq \pi_0(X,x) \iff C_H = \{\text{Paths } \gamma : [0,1] \longrightarrow X : \gamma(0) = x_0\}/\sim$

where $\gamma \sim \delta$ if $\delta * \bar{\gamma}$ is a loop based at e and $[\delta * \bar{\gamma}] \in H$.

<u>Propn</u> A connected covering space G' of a connected Lie group G is also a Lie group in such a way that the projection is a Lie group homomorphism.

<u>Pf</u> For some $H \in \pi_1(G)$, $G' = \{\gamma : [0,1] \longrightarrow G : \gamma(0) = e\}/\sim$ w/ $\sim$ as above.



Then by the same reasoning as for the fundamental group, pointwise multiplication on paths gives a well-defined multiplication on G'; i.e. if $\gamma \sim \delta$ and $\gamma' \sim \delta'$, then $[\delta * \bar{\gamma}] \in H$ and $[\delta' \circ \bar{\gamma}'] \in H$, we have $[\delta\delta' * \bar{\gamma}\bar{\gamma}'] \in H \Rightarrow \gamma\gamma' \sim \delta\delta'$. It's easy to check this is a group structure. For smoothness, given $[\gamma_1], [\gamma_2] \in G'$, $\pi([\gamma_i]) = g_i$, $g_1 g_2 = g$, let B be simply-connected nbhd of g and B_i simply-cld nbhds of g_i so that $B_1 \cdot B_2 \subseteq B$ (possible since multiplication in G is cts). There exist lifts. Let $B_i' = \{[\gamma_i * \ell_i] : \ell_i \text{ is entirely within } B_i\}$. Then $\pi|_{B_i'} : B_i' \longrightarrow B_i$ is a diffeomorphism. Same for $B' = \{[\gamma_1 \cdot \gamma_2 * \eta_i] : \eta_i \subseteq B\}$

$\uparrow$

$[\gamma_1 * \ell_1] \cdot [\gamma_2 * \ell_2] = [\gamma_1 \cdot \gamma_2 * \ell_1 \cdot \ell_2] \in B'$

Then we have

$$\begin{tikzcd} B_1' \times B_2' \arrow[r, "m"] \arrow[d, "\wr"] & B' \arrow[d, "\wr"] \\ B_1 \times B_2 \arrow[r, "m"] & B \end{tikzcd}$$

So multiplication is smooth.