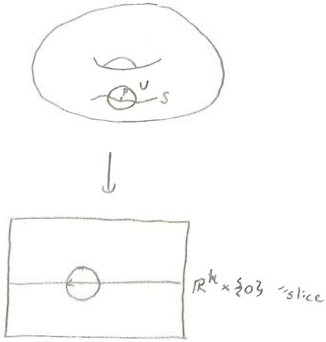
<u>Defn</u> A <u>regular submfd</u> of M of dimension k is a subset $S \subseteq M$ st every $p \in S$ has an "adapted chart", $\varphi : U \to V$ and st $\varphi(p) = 0$ and $\varphi(U \cap S) = (\mathbb{R}^k \times \{0\}) \cap V$.

<u>Note</u> Restricting adapted charts to S furnishes an atlas making S a smooth mfd of dimn k.

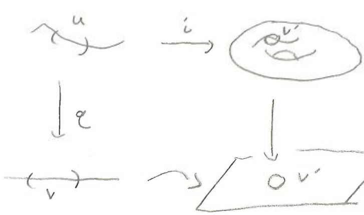


<u>Thm</u> $S \subseteq M$ regular submfds are precisely the subsets of M which are images of embeddings.

$\Rightarrow$ Regular submanifolds are embedded via the inclusion map

$\Leftarrow$ Suppose $i : S \to M$ is an embedding and $\varphi : \underset{S}{\overset{\cap}{U}} \to \underset{\mathbb{R}^k}{\overset{\cap}{V}}$ be a chart near $p \in S$.

Let $\psi : \underset{M}{\overset{\cap}{U'}} \to \underset{\mathbb{R}^n}{\overset{\cap}{V'}}$ be a chart near $i(p)$. We can choose the charts so that $i(S) \cap U' = i(U)$ by shrinking if necessary. Then $F : \psi \circ i \circ \varphi^{-1} : V \to V'$



has injective derivative at $\varphi(p)$.

Let $u_1, \ldots, u_{n-k}$ be a basis for $(\operatorname{im} D_{\varphi(p)} F)^{\perp}$ and let

$$\hat{F} : V \times \mathbb{R}^{n-k} \to V'$$
$$(\vec{x}, \vec{y}) = F(\vec{x}) + \sum y_i u_i$$

The inverse function theorem implies that $\hat{F}$ is a local diffeomorphism.

$$\begin{array}{ccc} (\varphi(p), 0) \in W & \longrightarrow & W' \\ \cap & & \cap| \\ V \times \mathbb{R}^{n-k} & & V' \end{array}$$

Then $\hat{F} \circ \psi$ provides an adapted chart near $i(p)$. This is a diffeomorphism onto its image: If $\pi : W \to V$ is the projection on the first factor, then the inverse to $i : S \to i(S)$ is given on charts by $\varphi^{-1} \circ \pi \circ \hat{F}^{-1} \circ \psi$, hence smooth, so $i : S \to i(S)$ is a diffeomorphism.

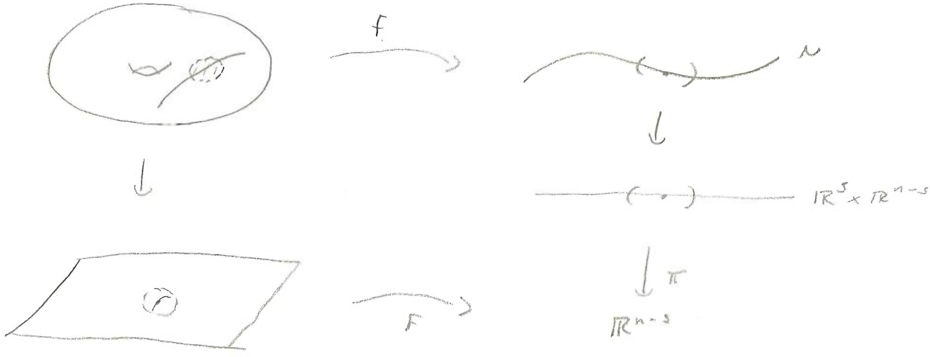
<u>Propn</u> $F : M \to N$ smooth, $S \subseteq N$ regular submfd consisting entirely of regular values of F, i.e. $D_x F : T_x M \to T_{F(x)} N$ surjective $\forall\, x \in M$ w/ $F(x) \in S$. Then $F^{-1}(S) \subseteq M$ is a regular submfd w/ dimn $\dim(M) - \dim(N) + \dim(S)$.

<u>PF</u> For $y \in S$, let $\psi : U \to V$ be an adapted chart, $\psi(y) = \vec{0}$.

$$\begin{array}{ccc} \cap| & & \cap| \\ N & & \mathbb{R}^n \end{array}$$

For $x \in F^{-1}(y)$, let $\varphi : U' \to V'$ be a chart,

$$\begin{array}{ccc} \cap| & & \cap| \\ M & & \mathbb{R}^m \end{array}$$



Let $F = \pi \circ \psi \circ f \circ \varphi^{-1}$ and let $u_1, \ldots, u_r$ be a basis for $\ker(D_{\varphi(x)} F)$ w/ $r = n-m-s$. Define $\hat{F} : V' \longrightarrow \mathbb{R}^m$ by $\tilde{F}(x) = (F(x), u_1 \cdot x, \ldots, u_r \cdot x)$

Then $\ker D_{\varphi(x)} \hat{F} = \ker D_{\varphi(x)} F \cap (\ker D_{\varphi(x)} F)^{\perp} = 0$. So $D_{\varphi(x)} \tilde{F}$ is an isomorphism.

Ergo $\hat{F}$ is a local diffeomorphism on a nbhd of $\varphi(x)$. $\hat{F} \circ \varphi$ then provides the adapted chart for $f^{-1}(s)$.

<u>Example</u> $\begin{cases} x^2+y^2+z^2 = 1 \\ x^3+y^3+z^3 = 0 \end{cases}$ defines a smooth 1-mfd in $\mathbb{R}^3$. For

$F : \mathbb{R}^3 \to \mathbb{R}^2$ w/ $F(x,y,z) = (x^2+y^2+z^2,\ x^3+y^3+z^3)$ has

$DF = \begin{pmatrix} 2x & 2y & 2z \\ 3x^2 & 3y^2 & 3z^2 \end{pmatrix}$ has rank < 2 only at $\{(t,t,t) : t \in \mathbb{R}\}$, which is

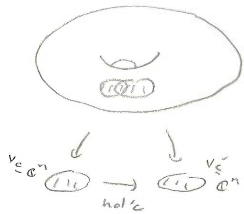
disjoint from $F^{-1}((1,0))$.

<u>Propn</u> If $f : M \to N$ is smooth, $M' \subseteq M$, $N' \subseteq N$ regular submanifold, and $f(M') \subseteq N'$, then $f|_{M'} : M' \longrightarrow N'$ is smooth.

<u>Pf</u> Obvious. But note, not true for merely immersed N'.

We also have a notion of <u>complex</u> manifolds.

- Replace $\mathbb{R}^n$ w/ $\mathbb{C}^n$ and smooth w/ holomorphic in the definition of a chart



- Natural notion of maps is holomorphic maps.
- This theory is much more rigid because hol'c maps are more restrictive

<u>Example</u>

- <u>Whitney</u> Every n-dim'l real mfd embeds in $\mathbb{R}^{2n}$
- It is relatively rare for complex mfds to embed into $\mathbb{C}^n$ (<u>Stein</u>)
  - Any hol'c fcn on a cpt complex ctd mfd is constant by Liouville's Thm.

$$M \hookrightarrow \mathbb{C}^n \xrightarrow{f_i} \mathbb{C} \quad \text{is nonconstant}$$

<u>Examples</u>

- $\mathbb{C}^n$
- $GL(n, \mathbb{C})$
- Oriented surfaces 

Defn G is a topological group if G is a topological space equipped w/ a group structure st

$$m: G\times G \to G \qquad i: G \to G$$
$$(g,h) \mapsto gh \qquad g \mapsto g^{-1}$$

are continuous maps.

Examples

1. Any G w/ the discrete topology.
2. Any G w/ the indiscrete topology (although we often restrict to Hausdorff & don't consider this
3. Given H any subgroup of G, the closure $\overline{H}$ as a subspace is also a subgroup, hence a topological group. (Exercise)
4. Given $H \trianglelefteq G$ closed, G/H inherits the structure of a topological group

We have homeomorphisms

$$L_g: G \to G \qquad R_g: G \to G$$
$$h \mapsto gh \qquad h \mapsto hg$$

and a cts map

$$G\times G \to G$$
$$(g,h) \mapsto ghg^{-1}$$

Defn A Lie group is a group G which is also a smooth mfd such that the group operations

$$\begin{array}{ll} G \times G \longrightarrow G & G \longrightarrow G \\ (g,h) \longmapsto gh & g \longmapsto g^{-1} \end{array} \quad \text{are smooth}$$

Eg

- $(\mathbb{R}, +)$
- $(\mathbb{R}^n, +)$
- $(\mathbb{Z}, +)$
- $\mathbb{R}^\times = (\mathbb{R} \setminus 0, \times)$
- $\mathbb{C}^\times = (\mathbb{C} \setminus 0, \times)$
- $GL(n, \mathbb{R}) = \{ A \in M_n(\mathbb{R}) : \det A = 0 \}$] Open submanifold of $\mathbb{R}^{n^2}$
  $\det : GL(n, \mathbb{R}) \longrightarrow \mathbb{R}$
- $GL(n, \mathbb{C}) \subseteq \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$
- $GL(n, \mathbb{H}) \subseteq \mathbb{H}^{n^2} = \mathbb{R}^{4n^2}$
- G, H Lie groups $\Rightarrow$ $G \times H$ is a Lie group.
- $S^1 \subseteq \mathbb{C}^\times$

$$\begin{tikzcd} S^1 \times S^1 \arrow[r] \arrow[d, phantom, "\cap" description] & S^1 \arrow[d, phantom, "\cap" description] \\ \mathbb{C}^\times \times \mathbb{C}^\times \arrow[r] & \mathbb{C}^\times \end{tikzcd}$$

and similarly for inverses
(restrictions of smooth fns to regular submfds are smooth)

- $T^n = (S^1)^n$ the n-torus is a Lie group.

- $SL(n,\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det A = 1\}$.

The map $\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is smooth, so to show $SL(n,\mathbb{R})$ is a smooth mfd it suffices to check that 1 is a regular value, ie. $D_A \det$ has rank 1 $\forall\ A \in SL(n,\mathbb{R})$.

But

$$\begin{tikzcd} SL(n,\mathbb{R}) \arrow[rr, "A"] \arrow[dr, "\det"'] & & SL(n,\mathbb{R}) \arrow[dl, "\det"] \\ & \mathbb{R} & \end{tikzcd}$$

is a diffeomorphism, so it suffices to check this at I. We can compute the directional derivative: $\det(I + tB) = \det\left(\begin{bmatrix} 1+tb_{11} & tb_{12} & & tb_{1n} \\ \vdots & & & \vdots \\ tb_{n1} & - & - & 1+tb_{nn} \end{bmatrix}\right)$

$$= 1 + t\sum b_{ii} + t^2(\ldots)$$

$\rightsquigarrow D_I \det = \sum b_{ii} = Tr(b) \neq 0$ is the directional derivative.

$\rightsquigarrow D \det$ has rank 1

- Likewise $SL(n,\mathbb{C})$, $SL(n,\mathbb{H})$

<u>Exercise</u> If $F : GL(n,\mathbb{R}) \longrightarrow \{B \in M_{n\times n}(\mathbb{R}) : B^T = B\}$

$$A \longmapsto A^T A$$

then I is a regular value and $O(n) = \{A : A^T A = I \text{ in } GL(n,\mathbb{R})\}$ is a Lie group. (Indeed, a compact Lie group – hw.)

<u>Note</u> $O(n) = \{A \in GL(n,\mathbb{R}) : \langle Av, Aw\rangle = \langle v, w\rangle$

<u>Likewise</u> $U(n) = \{A \in GL(n,\mathbb{C}) : \bar{A}^T A = I\}$

$Sp(n) = \{A \in GL(n,\mathbb{H}) : \bar{A}^T A = I\}$

conjugation changes the signs of i, j, and k.

Warning There is also a noncompact $Sp(n)$ in the literature.

Ex $O(1) \cong \{\pm 1\}$
$U(1) \cong S^1$
$Sp(1) \cong S^3$

Defn If G, H are Lie groups, a Lie group homomorphism $f: G \to H$ is a smooth map of manifolds which is also a group homomorphism.

eg $\cdot \det(GL(n,\mathbb{R})) \to \mathbb{R}^*$; $\det(O(n)) \to O(1)$

$\cdot$ Inclusions $O(n) \hookrightarrow GL(n,\mathbb{R})$

$\cdot$ Multiplication $G \times G \to G$ an an abelian group.

Defn The special orthogonal group $SO(n) = \{A \in O(n) : \det A = 1\}$

Exercise This is connected.

Exercise Check that open and closed subgroups of Lie groups are Lie groups (and in particular that $SO(n)$ is).

Exercise $SO(2) \cong S^1$ as Lie groups, via
$$f: S^1 \to SO(2)$$
$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Exercise If $f: G \to H$ is a group homomorphism and a diffeomorphism the inverse is also a group homomorphism.

Remark
$$i: \mathbb{R} \to T^2$$
$$t \mapsto (e^{it}, e^{\alpha it})$$
is an injective Lie group homomorphism which is not an embedding.

Propn G a Lie group, G_0 path connected cpt containing e. Then G_0 is an embedded Lie subgroup, and all cpts of G are diffeomorphic to G_0.

PF Let γ be a path from e to g, δ a path from e to h in G_0. Then $\gamma\cdot\delta$ pointwise is a path from e to gh. So G_0 is closed under multiplication. Likewise inversion. Hence G_0 is closed under group operations and is an embedded submfd $\Rightarrow$ G_0 is an embedded subgroup. Second statement follows since L_g is a diffeo from the cpt containing the identity to the cpt containing g.

eg $O(n)_{id} = SO(n)$

So we always have $1 \to G_0 \longrightarrow G \longrightarrow \underbrace{G/G_0}_{\text{discrete}} \longrightarrow 1$

Recall The fundamental group is the group of paths

$$\pi_0(X, x_0) = \left\{ \begin{array}{l} \gamma : [0,1] \to X \\ \gamma(0) = \gamma(1) = x_0 \end{array} \; : \; \gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \right\} \Big/ \sim$$

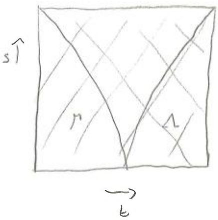
$\exists$ a cts map
$H : [0,1] \times [0,1] \to X$
$(s,t) \longmapsto h_s(t)$
$h_0 = \gamma_0$, $h_1 = \gamma_1$, $h_s(0) = x_0$

Propn G a connected Lie group $\Rightarrow$ $\pi_1(G)$ is abelian

PF γ, δ loops based at $e \in G$. Then $\gamma\delta := \gamma(t)\delta(t)$ is also a loop based at the identity, and if

$$\Gamma(s,t) = \begin{cases} \gamma((2-s)t) & \text{if } (2-s)t \le 1 \\ e & \text{if } (2-s)t \ge 1 \end{cases}$$

$$\Lambda = \begin{cases} \delta((2-s)t-(1-s)) & \text{if } (2-s)t-(1-s) \geq 1 \\ e & \text{if not} \end{cases}$$



Then $\Gamma\Delta(1,t) = \gamma(t)\delta(t)$

$\Gamma\Delta(0,t) = \gamma * \delta$ the concatenation of paths

So $\gamma * \delta \simeq \gamma\delta \sim \delta * \gamma$

$\Rightarrow [\gamma][\delta] = [\delta][\gamma]$ in $\pi_1(G)$. We also showed these are each equal to $[\gamma\delta]$.