(Review of) Manifolds

So far we've talked about finite and in particular discrete groups. But this class is actually about groups w/ some topology & in particular the structure of a manifold.

Let M be a topological space.

Recall

Defn M is Hausdorff if $\forall\ x \neq y \in M$, $\exists$ open $U, V \subseteq M$: $x \in U$, $y \in V$, $U \cap V = \emptyset$.

Defn A set of open subsets $\{U_i \subseteq M : i \in I\}$ is a basis for M if every open $U \subseteq M$ is $U = \bigcup_{i \in J} U_i$ for some $J \subseteq I$.

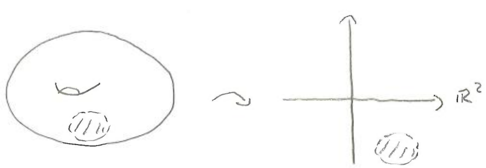
Defn A topological space is second-countable if it has a countable basis.

Ex $\{\prod_{i=1}^{n} (\alpha_i, \beta_i) : \alpha_i < \beta_i \in \mathbb{Q}\}$ is a countable basis for $\mathbb{R}^n$

Ex Any uncountable set w/ the discrete topology is not second countable.

Defn An n-dim'l chart for an open set $U \subseteq M$ is a homeomorphism $\varphi : U \to V$ where $V \subseteq \mathbb{R}^n$ is open.

M/

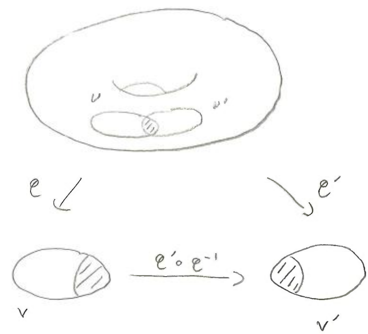


Defn M is locally Euclidean of dimension n if every point has a nbhd w/ an n dim'l chart.

Defn A topological manifold of dimn n is a Hausdorff second-countable locally Euclidean topological space.

Example $U \subseteq \mathbb{R}^n$ open, $F: U \to \mathbb{R}^k$ cts $\Rightarrow$ the graph $\Gamma(F)$ is defined to be $\{(x, f(x)) \in U \times \mathbb{R}^k : x \in U\}$ w/ a single chart given by projection.

Defn Two charts $\varphi: U \to V$ and $\varphi': U' \to V'$ are compatible if the overlap map $\varphi' \circ \varphi^{-1}: \varphi(U') \to \varphi'(U)$ is a smooth function.



Defn A (smooth) atlas $\mathcal{Q} = \{(U_i, \varphi_i) : i \in I\}$ for a topological manifold M consists of an open cover $M = \bigcup_{i \in I} U_i$ and charts $\varphi_i : U_i \xrightarrow{\sim} V_i \subseteq \mathbb{R}^n$ which are pairwise compatible.

Defn An atlas $\mathcal{Q}$ is maximal if every chart compatible with all the charts in $\mathcal{Q}$ is in $\mathcal{Q}$.

Exercise Every atlas is contained in a unique maximal atlas.

Defn A smooth structure on a topological mfd is a choice of maximal atlas. A smooth manifold is a topological mfd equipped w/ a smooth structure.

Example

1. Countable set w/ the discrete topology

2. $\mathbb{R}^n$. Maximal atlas: $\{\varphi : U \subseteq \mathbb{R}^n \longrightarrow V \subseteq \mathbb{R}^n$: smooth w/ smooth inverse$\}$

3. Any open $V \subseteq \mathbb{R}^n$

4. The graph of any smooth $F : U \longrightarrow \mathbb{R}^k$ via projection onto the first factor

5. The n-sphere $S^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1\}$

Charts $U_i^+ = \{(x_0, \dots, x_n) \in S^n : x_i > 0\}$ $2n+2$ charts $\quad \varphi_i^\pm : U_i^\pm \longrightarrow B_1(0) \subseteq \mathbb{R}^n$

$U_i^- = \{(x_0, \dots, x_n) \in S^n : x_i < 0\}$ $\quad (x_0, \dots, x_n) \longmapsto (x_0, \dots, \widehat{x_i}, \dots, x_n)$

6. $Mat_{n,k}(\mathbb{R}) \simeq \mathbb{R}^{nk}$

7. Products: For M smooth of dimension m and N smooth of dimension n, $M \times N$ is a smooth $m+n$ mfd.

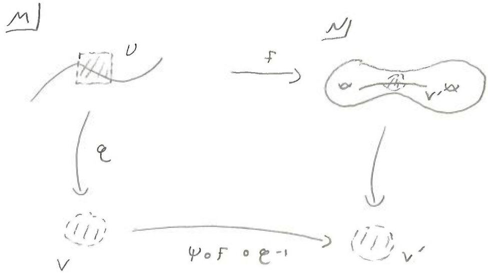
eg $T^n = \underbrace{S^1 \times \dots \times S^1}_{n}$

Smooth Functions M, N smooth manifolds of dimn m, n respectively.

Defn $F : M \to N$ is smooth if $\forall$ chart $\varphi : U \to V \subseteq \mathbb{R}^m$ on M,
$\psi : U' \to V' \subseteq \mathbb{R}^n$ on N

$\psi \circ F \circ \varphi^{-1}$ is smooth where defined, i.e. smooth as a function

$\underset{\subseteq \mathbb{R}^m}{\varphi(F^{-1}(U'))} \to \mathbb{R}^n$.



Ex If $M = \mathbb{R}^m$, $N = \mathbb{R}^n$, then f smooth in this sense $\Leftrightarrow$ f is smooth in the usual sense.

Ex f is smooth $\Leftrightarrow$ $\psi \circ f \circ \varphi^{-1}$ is smooth for all φ, ψ in a pair of atlases on M and N not necessarily maximal.

Ex Compositions of smooth maps are smooth.

Defn A diffeomorphism is a smooth homeomorphism w/ smooth inverse.

Example $\exp : \mathbb{R} \longrightarrow \mathbb{R}_+$

Remark
- Diffeomorphism is an equivalence relation.
- All manifolds of dimn ≤ 3 have unique-up-to-diffeomorphism smooth

Thm (Milnor, 1957) $\exists$ exactly 28 equivalence classes of smooth structure on S^7.

Thm (Donaldson & Taubes, 1970) $\exists$ uncountably many equivalence classes of smooth structure on $\mathbb{R}^4$.

Thm (Kervaire, 1960, later Donaldson) $\exists$ nonsmoothable mfds.

Derivations & tangent spaces

M a smooth mfd, $p \in M$

Defn $C^\infty(M) = \{\text{smooth fcns } f: M \to \mathbb{R}\}$. This is an $\mathbb{R}$-algebra.

Defn A derivation at p is a linear map $V: C^\infty(M) \to \mathbb{R}$ st

$V(fg) = V(f)g(p) + f(p)V(g)$.

Ex $M = \mathbb{R}^n$, $V(f) = \frac{\partial f}{\partial x_i}\Big|_p$

Ex Directional derivatives along a parametric curve γ.

$$\gamma: (-\varepsilon, \varepsilon) \to M \text{ smooth}$$

$$\gamma(0) = p$$

$V(f) = (f \circ \gamma)'(0)$

Defn The tangent space T_pM to M at p is the vector space of all derivations at p.

Propn Any derivation may be expressed as a directional derivative.

Corollary Since $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ is a basis for $T_0\mathbb{R}^n$, T_pM is n-dimensional.

Defn If $\varphi: M \to N$ is smooth, then the push-forward or derivative at $p \in M$ is the linear map

$$\varphi_* = D_p\varphi : T_pM \longrightarrow T_{\varphi(p)}N$$

given by $\varphi_* V(f) = V(f \circ \varphi)$.

Exercise This is a derivation.

Exercises (a) There is a chain rule: $M \xrightarrow{\psi} N \xrightarrow{\varphi} P$ has

$$(\varphi\circ\psi)_* = \varphi_* \circ \psi_* \quad \text{i.e.} \quad D_p(\varphi\circ\psi) = D_{\psi(p)}(\varphi)\circ D_p\psi$$

(b) φ a diffeomorphism $\Rightarrow$ φ_* is an isomorphism and $(\varphi_*)^{-1} = (\varphi^{-1})_*$.

Example $\varphi : \mathbb{R}^m \to \mathbb{R}^n$. Then $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$

$$V = \frac{\partial}{\partial x_j} \quad f(\vec{y}) = y_i \quad (\varphi_* V)(f) = \frac{\partial}{\partial x_i}(y_i \circ \varphi)_p = \frac{\partial \varphi_i}{\partial x_j}\, p$$

So in terms of the bases $\{\frac{\partial}{\partial x_i}\}$ and $\{\frac{\partial}{\partial y_i}\}$, the matrix of φ_* is given by the Jacobean.

After choosing a chart, $T_pM \simeq T_{\varphi(p)}$, $\mathbb{R}^n \simeq \mathbb{R}^n$ noncanonically. Ergo the rank of the derivative at p is well-defined.

Submersions, immersions, and embeddings

$f: M \to N$ smooth

Defn f is a submersion if its derivative is surjective for all $x \in M$ and an immersion if it is injective.

eg $\mathbb{R}^k \times \{0\} \hookrightarrow \mathbb{R}^m$ immersion

$\mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ submersion

More generally $M\times N \xrightarrow{\pi} M$ submersion $\quad D_{(p,q)}(\pi) : T_{(p,q)}(M\times N) \twoheadrightarrow T_pM$

$$T_{(p,q)}(M\times N) \cong T_pM \times T_qN$$

Ergo parametric curves and surfaces in $\mathbb{R}^3$ are immersed

- $(x(t), y(t), z(t))$: $\forall$ $t \in (a,b)$, $\underbrace{(x'(t), y'(t), z'(t))}_{\parallel \; e_*\left(\frac{d}{dt}\right)} \neq (0,0,0)$

- $S : (a,b) \times (c,d) \longrightarrow \mathbb{R}^3$ $\quad 0 \neq \vec{n} = \frac{dS}{du} \times \frac{dS}{dv} \iff \left.\begin{array}{l} \frac{dS}{du} = S_* \frac{d}{du} \\ \frac{dS}{dv} = S_* \frac{d}{dv} \end{array}\right\}$ lin indep

IF $F : M \longrightarrow N$, $F_* = D_pF$ is injective/surjective $\iff (\psi \circ F \circ \varphi^{-1})_*$ is injective/surjective.

Charts

$(\psi \circ F \circ \varphi^{-1})_* = \psi_* \circ F_* \circ \varphi_*^{-1}$

Isos

So one can check immersion/submersion on charts.

Embedding

Defn An injective immersion $F : M \longrightarrow N$ is an embedding if it is also a homeomorphism onto its image.

eg ① Inclusion $\mathbb{R}^k \times \{0\} \hookrightarrow \mathbb{R}^n$

② Inclusion of an open $U \subseteq M$

③ Graph of a smooth Fcn.

Counterexamples (Immersion but not embedding)

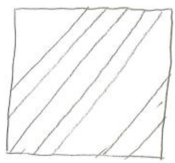
①  $\subseteq \mathbb{R}^2$

②  $\subseteq \mathbb{R}^2$ Not a homeomorphism onto its image

③ Line of irrational slope on a torus

$S^1 = \{z \in \mathbb{C} : |z| = 1\}$ $\quad \alpha \in \mathbb{R} \setminus \mathbb{Q}$ $\quad f : \mathbb{R} \to T = S^1 \times S^1$

$t \mapsto (e^{it}, e^{\alpha i t})$



Exercise: f is smooth but the subspace topology on this line is not the subspace topology on the reals.

Inverse function theorem

Thm For any $p \in U$ open in $\mathbb{R}^n$, $f : U \to \mathbb{R}^n$ smooth, $D_p f$ is an isomorphism $\iff$ an inverse to f exists near p, ie $\exists$ a neighborhood $V \ni p$: $f(V)$ open, $f|_V : V \to f(V)$ has a smooth inverse.

Corollary $f : U \to \mathbb{R}^m$ smooth st $D_p f$ injective, then f is an embedding

$U \subseteq \mathbb{R}^n$

near p, ie $\exists$ a nbhd $V \ni p$ st $f|_V$ is an embedding.

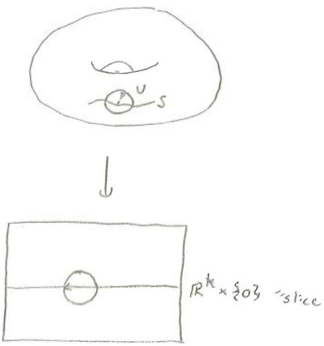
] Restriction to V is an embedding.

PF Let $u_1, \dots, u_{m-n} \in \mathbb{R}^m$ be a basis for $(\operatorname{im} D_p f)^\perp$ and define $\tilde{f} : U \times \mathbb{R}^{m-n} \to \mathbb{R}^m$ by $\tilde{f}(\vec{x}, \vec{y}) = f(\vec{x}) + \sum y_i u_i$. Then $\operatorname{im} D_{(p,0)} \tilde{f} = \operatorname{im} D_p f + (\operatorname{im} D_p f)^\perp = \mathbb{R}^m$

Then $D_{(p,0)} \tilde{f}$ is an isomorphism. We apply the inverse ftn thm to $\tilde{f}$ to get an inverse for $\tilde{f}$ near $(p, 0)$. Ergo $f_{V \cap (U \times \{0\})} = \tilde{f}|_{V \cap (U \times \{0\})}$ has injective derivative and is a homeomorphism onto its image. Hence $f|_{V \cap (U \times \{0\})}$ is an embedding.

<u>Defn</u> A <u>regular submfd</u> of M of dimension k is a subset $S \subseteq M$ st every $p \in S$ has an "adapted chart", $\varphi : U \to V$ and st $\varphi(p) = 0$ and $\varphi(U \cap S) = (\mathbb{R}^k \times \{0\}) \cap V$.

<u>Note</u> Restricting adapted charts to S furnishes an atlas making S a smooth mfd of dimn k.

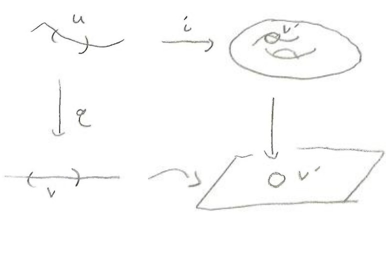


<u>Thm</u> $S \subseteq M$ regular submfds are precisely the subsets of M which are images of embeddings.

$\Rightarrow$ Regular submanifolds are embedded via the inclusion map

$\Leftarrow$ Suppose $i : S \to M$ is an embedding and $\varphi : U \to V$ be a chart near $p \in S$. $U \subseteq S$, $V \subseteq \mathbb{R}^k$.

Let $\psi : U' \to V'$ be a chart near $i(p)$. $U' \subseteq M$, $V' \subseteq \mathbb{R}^n$. We can choose the charts so that $i(S) \cap U' = i(U)$ by shrinking if necessary. Then $f : \psi \circ i \circ \varphi^{-1} : V \to V'$



has injective derivative at $\varphi(p)$.

Let $u_1, \dots, u_{n-k}$ be a basis for $(\operatorname{im} D_{\varphi(p)} f)^{\perp}$ and let

$$\hat{F} : V \times \mathbb{R}^{n-k} \to V'$$
$$(\vec{x}, \vec{y}) \mapsto f(\vec{x}) + \sum y_i u_i$$

The inverse function theorem implies that $\hat{F}$ is a local diffeomorphism.

$$\begin{array}{ccc} (\varphi(p), 0) \in W & \longrightarrow & W' \\ \cap & & \cap \\ V \times \mathbb{R}^{n-k} & & V' \end{array}$$

Then $\hat{F} \circ \psi$ provides an adapted chart near $i(p)$. This is a diffeomorphism onto its image: if $\pi : W \to V$ is the projection on the first factor, then the inverse to $i : S \to i(S)$ is given on charts by $\varphi^{-1} \circ \pi \circ \hat{F}^{-1} \circ \psi$, hence smooth, so $i : S \to i(S)$ is a diffeomorphism.

<u>Propn</u> $F : M \to N$ smooth, $S \subseteq N$ regular submfd consisting entirely of regular values of F, i.e. $D_x F : T_x M \to T_{F(x)} N$ surjective $\forall\, x \in M$ w/ $F(x) \in S$. Then $F^{-1}(S) \subseteq M$ is a regular submfd w/ dimn $\dim(M) - \dim(N) + \dim(S)$.

<u>PF</u> For $y \in S$, let $\begin{array}{ccc} \psi : U & \to & V \\ \cap & & \cap \\ N & & \mathbb{R}^n \end{array}$ be an adapted chart, $\psi(y) = \vec{0}$.

For $x \in F^{-1}(y)$, let $\begin{array}{ccc} \varphi : U' & \to & V' \\ \cap & & \cap \\ M & & \mathbb{R}^m \end{array}$ be a chart