The center of $\mathbb{k}[G]$ — G finite, $\mathbb{k}$ algebraically closed

Recall from last time: $\mathbb{k}[G] \cong M_{n_1}(\mathbb{k}) \times \cdots \times M_{n_r}(\mathbb{k})$ where $n_1, \ldots, n_r$ are the dimns of the irreps of G

Let $Z(G)$ is the center of $\mathbb{k}[G]$. This is naturally identified w/ $C(G)$ the class functions, since $Z(G) = \{ \sum_{g \in G} \varphi(g) g : \varphi(g) = \varphi(hgh^{-1}) \}$

Also identified w/ $\mathbb{C}e_1 \times \cdots \times \mathbb{C}e_r \subseteq M_{n_1}(\mathbb{k}) \times \cdots \times M_{n_r}(\mathbb{k})$, with $\begin{cases} e_{ij} = \delta_{ij} e_i \\ 1_G = e_1 + \cdots + e_r \end{cases}$

If $\rho_j : G \to GL(V_j)$ is irreducible e_j acts on V_j via identity and $\rho_j \circ e_i = \delta_{ij} \, Id_{V_j}$ $(*)$

Lemma If χ_i is the character of the irrep ρ_i of dimn n_i, then

$$e_i = \underbrace{\frac{n_i}{|G|} \sum_{g \in G} \chi_i(g^{-1}) g}_{f_i}$$

Pf Suffices to check we have written down the operator for which $(*)$ holds. Certainly since $\rho_j \circ f_i \in End(V_j)$, Schur's Lemma implies it is $\lambda\, Id$ for some λ. Then $n_j \lambda = tr(\rho_j \circ f_i) = \frac{n_i}{|G|} \sum \chi_i(g^{-1}) \chi_j(g) = \frac{n_i}{|G|} \langle \chi_i, \chi_j \rangle = \delta_{ij} n_i$.
So $n_j \lambda = \delta_{ij} n_i \Rightarrow \lambda = \delta_{ij}$. Hence $f_i = e_i$. $\square$

Remark A careful analysis of $Z(G)$ can be used to show that the order of an irrep ρ of G divides $|G|$.

Induction & Restriction

Say G is finite & H is a subgroup. Then we have $\mathbb{C}[H] \subseteq \mathbb{C}[G]$.

Let V be a representation of H; then $\operatorname{Ind} V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$

If $G = \coprod_{i=1}^{k} g_i H$ as cosets, so that $k = [G:H]$, then $\dim(\operatorname{Ind}(V)) = k \dim V$.

Action looks like $g(g_i \otimes v) = gg_i \otimes v = g_j h \otimes v = g_i \otimes hv$

Examples ① $V = \mathbb{C}[H]$ regular representation;

$$\operatorname{Ind} V = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}[H] = \mathbb{C}[G]$$

② V is the trivial rep. $\operatorname{Ind} V$ has a basis $\{g_i \otimes 1\}$ w/ action $gg_i \otimes 1 = g_j h \otimes 1 = g_j \otimes h$] Action of G on cosets by left multiplication.

This is a special case of A, B rings, $B \subseteq A$

$$\begin{tikzcd} B\text{-modules} \arrow[r, bend left, "\operatorname{Ind}"] & A\text{-modules} \arrow[l, bend left, "\operatorname{Res}"] \end{tikzcd}$$

$$V \longmapsto A \otimes_B V$$

$${}_B W \longleftarrow\!\shortmid\; W$$

(as a B-module)

Induction is left-adjoint to restriction, so $\operatorname{Hom}_A(A \otimes_B V, W) = \operatorname{Hom}_B(V, {}_B W)$

Ergo $\mathrm{Hom}_G(\mathrm{Ind}\,V, W) \simeq \mathrm{Hom}_H(V, \mathrm{Res}\,W)$

If V and W are irreducible, this implies the dimn of W in $\mathrm{Ind}\,V$ and the dimn of V in $\mathrm{Res}\,W$ are equal.

Consequences If H is abelian, $\mathbb{C}[H] = \bigoplus_{i=1}^{|H|} V_i$ ← one-dim'l irreps

Then $\mathrm{Ind}(\mathbb{C}[H]) = \bigoplus_{i=1}^{|H|} \underbrace{\mathrm{Ind}(V_i)}_{\dim [G:H]} = \mathbb{C}[G]$

So if W is an irrep of G, $W \subseteq \mathrm{Ind}(V_i)$ for some $V_i \Rightarrow W$ has dimn at most $[G:H]$.

Corollary An irrep of a dihedral or binary dihedral group is at most two dimensional.

Pf

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & D_n \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] \arrow[l, bend left] & 1 \\
1 \arrow[r] & \mathbb{Z}/2n\mathbb{Z} \arrow[r] & D_n^* \arrow[r] \arrow[u] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 1 \\
 & & \mathbb{Z}/2\mathbb{Z} \arrow[u] & &
\end{tikzcd}$$

Split

Nonsplit ($\mathbb{Z}/2\mathbb{Z}$ comes from an element of order 4)

$$\begin{tikzcd}
D_n^* \arrow[d] & \subseteq & SU(2) \arrow[d] \\
D_n \arrow[rr] & & SO(3)
\end{tikzcd}$$

$D_n = \{r, s : r^2 = s^n = Id,\ sr = rs^{-1}\}$

$D_n^* = \{\underset{g_2}{x}, \underset{g_2^{-1}g_1}{y}, \underset{g_1}{z} : x^2 = y^2 = z^n = xyz\}$ generated by

$$\begin{array}{cc} g_1 & g_2 \\ \begin{pmatrix} e^{\frac{2\pi i}{2n}} & 0 \\ 0 & e^{\frac{2\pi i}{2n}} \end{pmatrix} & \begin{pmatrix} 0 & e^{\frac{2\pi i}{4}} \\ e^{\frac{2\pi i}{4}} & 0 \end{pmatrix} \end{array}$$

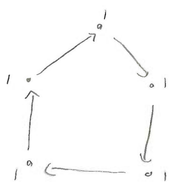
McKay graphs

G finite, $\rho: G \longrightarrow GL(V)$ a representation.

Defn The McKay graph of (G, ρ_0) is defined as follows:

- Vertices correspond to the irreps of G and are labeled by their dimension.
- There is a directed edge from ρ_i to ρ_j labelled w/ m_{ij} if ρ_j appears as a direct summand of $\rho_0 \rho_i$.

Example $G = \mathbb{Z}/n\mathbb{Z}$, ρ is a nontrivial representation corresponding to $e^{\frac{2\pi i}{n}}$.



Conventional to drop the 1's on arrows. We would also erase the directionality if there is an arrow in both directions.

Exercise V is faithful $\Leftrightarrow$ Γ is connected.

Application to the subgroups of $SU(2)$

We have

$$\begin{array}{ccc} SU(2) & & S^3 \\ \downarrow{\scriptstyle 2:1} & & \downarrow \\ SO(3) & & \mathbb{RP}^3 \end{array}$$

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} : a\bar{a} + b\bar{b} = 1 \right\}$$

Transformations of $\mathbb{C}^2$ preserving the Hermitian inner product & having determinant one.
$A\bar{A}^T = Id$, $\det A = 1$

$SU(2) \simeq$ Unit quaternions $\{a_1 + a_2 i + b_1 j + b_2 k\}$

$$a_1 + a_2 i + b_1 j + b_2 k$$
$$\downarrow$$
Rotation around (a_2, b_1, b_2) by 2θ w/ $\cos\theta = a_1$, $|\sin\theta| = \|(a_2, b_1, b_2)\|$

$$SO(3) = \{ B \in GL(\mathbb{R}^3) : BB^t = Id,\ \det B = 1 \}$$

Kernel is the unique element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = z$

The finite subgroups of $SO(3)$ are the symmetry groups of the regular polyhedra

- $\mathbb{Z}/n\mathbb{Z}$
- D_n
- A_4 (regular tetrahedron)
- S_4 (regular octahedron, cube)
- A_5 (dodecahedron/icosahedron)

So subgroups of $SU(2)$ are

- Odd order and isomorphic to a subgroup of $SO(3)$

  $\rightsquigarrow \mathbb{Z}/n\mathbb{Z}$ for n odd

- Even order and isomorphic to an extension of a subgroup of $SO(3)$

  $\rightsquigarrow \mathbb{Z}/n\mathbb{Z}$ for n even

  D_n^* binary dihedral

  A_4^* binary tetrahedral

  S_4^* binary octahedral

  S_5^* binary icosahedral

The McKay graph of a subgroup $G \subseteq SU(2)$ is the McKay graph associated to the canonical representation on $\mathbb{C}^2$ given by inclusion

Example $\mathbb{Z}/n\mathbb{Z} \longrightarrow \left\langle \begin{pmatrix} e^{\frac{2\pi i}{n}} & 0 \\ 0 & e^{-\frac{2\pi i}{n}} \end{pmatrix} \right\rangle$

This is the direct sum of a generator of the group of one-dim'l irreps and its dual

$\}$ n-1 vertices

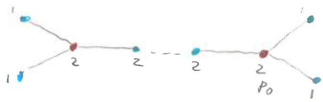
For a noncyclic group the canonical rep is always irreducible

<u>Thm</u> (McKay) Let G be a finite subgroup of $SU(2)$ and ρ_0 its canonical rep. Then the McKay graphs are

- $G = \mathbb{Z}/n\mathbb{Z}$, $|G| = n$

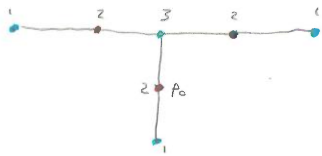
$\}_{n-1}$ $\hat{A}_{n-1}$

- $G = D_n^*$, $n \geq 2$, $|G| = 4n$



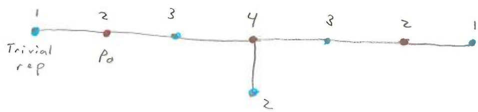
$2n-3$ 2-dim'l reps $\tilde{D}_n$

- A_4^*, $|A_4^*| = 24$



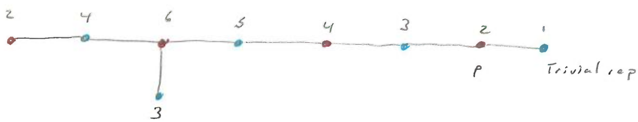
$\tilde{E}_6$

- S_4^*, $|S_4^*| = 48$



$\tilde{E}_7$

- A_5^*, $|A_5^*| = 120$



$\tilde{E}_8$

We can partition the irreps by whether z acts by ± 1. $V_i \subseteq V \otimes V_j$

$1 = -1 \otimes -1$

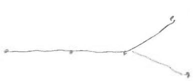
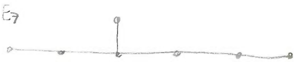
$-1 = -1 \otimes 1$

Exercise These are the only connected graphs w/ a weight fcn st $2d_i = \sum_{i - j} d_j$.

These are the affine (or extended) simply-laced Dynkin diagrams.

On any simple graph Γ, we can take a vector space $\mathbb{R}^n$ w/ basis $\{e_i\}_{i \text{ is a vertex}}$ and a product $(e_i, e_j) = \begin{cases} 2 & i = j \\ -1 & i - j \\ 0 & \text{otherwise} \end{cases}$

Thm Any connected simple graph is one of

Simply-laced Dynkin $(,)$ is pos. def.	Mackay/affine $(,)$ is pos. semidefine w/ $w_0 = \sum d_i e_i$ spans the null space	Indefinite
A_n 	$\tilde{A}_n$	Everything else
D_n 	$\tilde{D}_n$	
E_6 	$\tilde{E}_6$	
E_7 	$\tilde{E}_7$	
E_8 	$\tilde{E}_8$	

Example $G = A_4^*$ $\quad |A_4^*| = 24$ $\qquad$ **Presentation** $\langle s, t : (st)^2 = s^3 = t^3 \rangle$

This is a (nonsplit) extension $\quad 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow A_4^* \longrightarrow A_4 \longrightarrow 0$

$G \longrightarrow A_4 \longrightarrow A_4/[A_4, A_4] \cong \mathbb{Z}/3\mathbb{Z} \longrightarrow \mathbb{C}^*$] Three irreps of dimn 1

			(order 4)	(order 3)	(order 3)	(order 6)	(order 6)
	1	1	6	4	4	4	4
	1	-1	g_1	g_3	g_3^2	$-g_3$	$-g_3^2$
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
χ_3	1	1	1	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
χ_4	2	-2	0	-1	-1	1	1
χ_5	2	-2	0	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$
χ_6	2	-2	0	$e^{\frac{2\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$	$e^{\frac{4\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$
χ_7	3	3	-1	0	0	0	0

canonical rep

$\mathbb{R}^3$ via rotational symmetries of the tetrahedron

We get the graph by multiplying χ_4 w/ the other characters and decomposing into class functions.

eg $\chi_4 \cdot \chi_5 = \chi_2 + \chi_7$