Lecture 26

Root Systems - The General Case

Recall G a compact connected Lie group, T a maximal torus, $\dim(T) = r$. If $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, a root of G with respect to T is a nonzero weight of the adjoint representation (i.e., a character in the decomposition of $\mathfrak{p}_{\mathbb{C}}$ into $\mathfrak{t}$-invariant one-dim'l cpx subspaces wrt the adjoint representation).

Example $G = SU(2)$

$$\mathfrak{g}_{\mathbb{C}} = \Big\langle \underbrace{\overset{H_\alpha}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}}_{\substack{\uparrow \\ \text{spans } \mathfrak{t}_{\mathbb{C}}}}, \underbrace{\overset{X_\alpha}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \overset{X_{-\alpha}}{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}}_{\text{spans } \mathfrak{p}_{\mathbb{C}}} \Big\rangle \qquad Ad\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} X_\alpha = t^2 X_\alpha$$

$\rightsquigarrow$ X_α spans a $\mathfrak{t}$-invt vector space corresponding to the character λ_2

$$\lambda_{-2} \longleftarrow \circ \longrightarrow \lambda_2$$

General Case

Propn Let G, T as above. Let $\mathfrak{h}$ be a maximal abelian subgroup $\mathfrak{h}$ of $\mathfrak{g}$ is the Lie algebra of a conjugate of T. Its dimension is the rank r of G.

Pf Recall that $\exp(\mathfrak{h})$ is commutative (elements w/ zero bracket exponentiate to commuting elements), and connected since it is the image of a connected space. Its closure is a closed abelian subgroup of G, hence a Lie subgroup, hence a torus H. By maximality of H it must actually be a maximal torus. And then by Cartan's thm H and T are conjugate.

Lemma Suppose G is a compact Lie group w/ Lie algebra $\mathfrak{g}$, $\rho: G \to GL(V)$ a representation, and $d\rho: \mathfrak{g} \to \mathrm{End}(V)$ the differential. If $v \in V$ and $X \in \mathfrak{g}$ st $d\rho(X)^n v = 0$ for any $n > 1$, then $d\rho(X)v = 0$.

Pf We put a G-inv't positive definite inner product $\langle , \rangle$ on V. The inner product is then $\mathfrak{g}$-inv't, which means $\langle d\rho(X)v, w\rangle = -\langle v, d\rho(X)w\rangle$. Ergo $d\rho(X)$ is skew-Hermitian $\Rightarrow$ by the spectral theorem, V has a basis with respect to which the matrix of $d\rho(X)$ is diagonal. But for a diagonal matrix, $M^n v = 0 \Leftrightarrow Mv = 0$.

Let (ρ, V) be any finite dim'l cpx representation of G. If $\lambda \in X^*(T)$, let $V(\lambda) = \{v \in V : \rho(t)v = \lambda(t)v\}$, so that $V = \bigoplus_\lambda V(\lambda)$. Note that in the event that $(\rho, V) = (\mathrm{Ad}, \mathfrak{g}_{\mathbb{C}})$ and $\lambda = \alpha$ is a root, $V(\lambda) = \mathfrak{X}_\alpha$.

Propn Let (ρ, V) be an irrep of G, and let α be a root.

(i) If $d\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ is the differential of ρ, then $d\rho(H)v = d\lambda(H)v$ for all $H \in \mathfrak{t}$ and $v \in V(\lambda)$.

(ii) We have $[H, X_\alpha] = \mathrm{ad}(H)X_\alpha = d\alpha(H)X_\alpha$ for $H \in \mathfrak{t}$, $X_\alpha \in \mathfrak{X}_\alpha$.

(iii) If (ρ, V) is a finite-dim'l cpx representation of G and $v \in V(\lambda)$ for some $\lambda \in X^*(T)$, then $d\rho(X_\alpha)v \in V(\lambda + \alpha)$

PF For part (i), let $H \in \mathfrak{t}$ and $t \in \mathbb{R}$. Then for any $v \in V(\lambda)$, we see that $\exp(tH)$ lies in T, so by definition of $V(\lambda)$, we see that $\rho(\exp(tH))v = \lambda(\exp(tH))v$. So on $V(\lambda)$, $\rho(\exp(tH)) = \lambda(\exp(tH)) \Rightarrow$

$$\frac{d}{dt}\left(\rho(\exp(tH))\right)\Big|_{t=0} = \frac{d}{dt}\left(\lambda(\exp(tH))\right) \text{ on } V(\lambda) \Rightarrow d\rho(H)v = d\lambda(H)v \text{ for } v \in V(\lambda).$$

Since the differential of Ad is ad, part (ii) is then a special case of part (i) in the case that $\rho = \mathrm{Ad}$, $d\rho(H)(X_\alpha) = \mathrm{ad}(H)X_\alpha = [H, X_\alpha]$, and $\lambda = \alpha$.

For part (iii), we see that part (ii) implies that

$$d\rho(H)\,d\rho(X_\alpha) - d\rho(X_\alpha)\,d\rho(H) = d\rho[H, X_\alpha] = d\rho(d\alpha(H)\cdot X_\alpha) = d\alpha(H)\,d\rho(X_\alpha)$$

Always true

(ii)

This is a complex number

Now apply this to v. Let $w = d\rho(X_\alpha)v$.

$$\begin{aligned}
\underbrace{d\rho(H)\,d\rho(X_\alpha)v} &= d\rho(X_\alpha)\,d\rho(H)v + d\alpha(H)\cdot d\rho(X_\alpha)v \\
&\qquad (i) \\
d\rho(H)w &= d\rho(X_\alpha)\,d\lambda(H)(v) + d\alpha(H)w \\
d\rho(H)w &= d\lambda(H)\,d\rho(X_\alpha)v + d\alpha(H)w \\
d\rho(H)w &= (d\lambda(H) + d\alpha(H))(w)
\end{aligned}$$

$\Rightarrow w \in V(\lambda + \alpha)$.

With this in mind we can study the spaces $\mathfrak{X}_\alpha$ for the adjoint representation a little more carefully.

Notation Write $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$. Let $c : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ be the conjugation

$$X + iY \longmapsto X - iY$$

map, which is an automorphism of $\mathfrak{g}_{\mathbb{C}}$ as a real Lie algebra (although not a complex-linear automorphism. Note that $c(a z) = \bar{a}\, c(z)$. Now let G cpt, T a maximal torus, weight spaces as previously.

Propn (i) $c(\mathfrak{X}_\alpha) = \mathfrak{X}_{-\alpha}$.

(ii) If $X_\alpha \in \mathfrak{X}_\alpha$ and $X_\beta \in \mathfrak{X}_\beta$, and $\alpha, \beta \in \Phi$, then

$$[X_\alpha, X_\beta] \in \begin{cases} \mathfrak{t}_{\mathbb{C}} & \text{if } \beta = -\alpha \\ \mathfrak{X}_{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \end{cases}$$

while $[X_\alpha, X_\beta] = 0$ if $\beta \neq -\alpha$ and $\alpha + \beta \notin \Phi$.

(iii) If $0 \neq X_\alpha \in \mathfrak{X}_\alpha$, then $[X_\alpha, c(X_\alpha)]$ is a nonzero element of $i\mathfrak{t}$, and $d\alpha([X_\alpha, c(X_\alpha)]) \neq 0$.

In case (ii) if $\alpha + \beta \in \Phi$, we will eventually show that $[X_\alpha, X_\beta]$ is a nonzero element of $\mathfrak{X}_{\alpha+\beta}$.

Pf For part (i), recall that for $H \in \mathfrak{t}$, $d\alpha(H)$ takes purely imaginary values. Then since we have that

$$[H, X_\alpha] = d\alpha(H) X_\alpha$$

applying c we obtain

$$[H, c(X_\alpha)] = [c(H), c(X_\alpha)] = c[H, X_\alpha] = c(d\alpha(H) X_\alpha) = -d\alpha(H)\, c(X_\alpha)$$

$\Rightarrow c(X_\alpha) \in \mathfrak{X}_{-\alpha}$.

. Part (ii) is a special case of the last part of the preceding proposition: If $v \in V(\lambda)$, then $d\rho(x_\alpha)v_\lambda \in V(\lambda+\alpha)$. In this case $\rho = Ad$, $d\rho = ad$, so

$$[x_\alpha, x_\beta] \in \begin{cases} t_{\mathbb{C}} = V(0) & \text{if } \beta = -\alpha \\ \mathfrak{X}_{\alpha+\beta} = V(\alpha+\beta) & \text{if } \alpha+\beta \in \Phi \end{cases}$$

If $\alpha+\beta \neq 0$ and not in Φ, $V(\alpha+\beta) = \{0\}$ and we conclude that $ad(x_\alpha)x_\beta = [x_\alpha, x_\beta] = 0$ as well.

Now for part (iii). Parts (i) and (ii) imply that $[x_\alpha, \underset{\text{Lies in } \mathfrak{X}_{-\alpha}}{c(x_\alpha)}] \in t_{\mathbb{C}}$. Now apply c to this element to obtain

$$c([x_\alpha, c(x_\alpha)]) = [c(x_\alpha), x_\alpha] = -[x_\alpha, c(x_\alpha)],$$

so $[x_\alpha, c(x_\alpha)] \in it$. We claim that $[x_\alpha, c(x_\alpha)] \neq 0$. Let $t_\alpha \subseteq t$ be the kernel of $d\alpha$, which is of course a subspace of codimn 1. Let $H_1, \ldots, H_{r-1}$ be a basis thereof. If $[x_\alpha, c(x_\alpha)] = 0$, then if we let

$$Y_\alpha = \frac{1}{2}(x_\alpha + c(x_\alpha)) \quad \text{and} \quad Z_\alpha = \frac{1}{2i}(x_\alpha - c(x_\alpha))$$

then Y_α and Z_α are c-invt and therefore both lie in $\mathfrak{g}$, so we see that $H_1, \ldots, H_{r-1}, Y_\alpha, Z_\alpha$ are $r+1$ commuting elements in $\mathfrak{g}$ which are linearly independent over $\mathbb{R}$: Y_α and Z_α commute w/ each other, and $[H_i, x_\alpha] = d\alpha(H_i)x_\alpha = 0$, likewise $[H_i, c(x_\alpha)] = d(-\alpha)(H)(c(x_\alpha)) = 0$ since the H_i are also a basis for $\ker(d(-\alpha))$. But this is a contradiction; there is no abelian subalgebra of $\mathfrak{g}$ w/ dimn $r+1$. Ergo $[x_\alpha, c(x_\alpha)] \neq 0$.

We would furthermore like to show that $d\alpha([X_\alpha, c(X_\alpha)]) \neq 0$. Suppose for the sake of contradiction that it vanishes. Then if $H_0 = -i[X_\alpha, c(X_\alpha)] \in \mathfrak{t}$, we have

$$[H_0, X_\alpha] = d\alpha(H_0) X_\alpha = d\alpha(-i[X_\alpha, c(X_\alpha)]) X_\alpha = 0$$

and likewise $[H_0, c(X_\alpha)] = 0$. Then w/ Y_α and Z_α the same elements of $\mathfrak{g}$ as previously, we have $[H_0, Y_\alpha] = [H_0, Z_\alpha] = 0$. But a quick computation shows that $[Y_\alpha, Z_\alpha] = \frac{1}{2} H_0$. So we have both $[Y_\alpha, Z_\alpha] = \frac{1}{2} H_0$ and $[Y_\alpha, H_0] = 0$, so we see that $\mathrm{ad}(Y_\alpha)^2 Z_\alpha = 0$ and $\mathrm{ad}(Y_\alpha) Z_\alpha \neq 0$. But today's second lemma said that was impossible, so we are done.

We're now ready for our first geometric consequence!

<u>Propn</u> If $\dim(T) = 1$, then either $G = T$ or $\dim(G) = 3$. If α is any root, $\mathfrak{X}_\alpha$ is one dimensional, and the only other root is $-\alpha$.

<u>PF</u> Since $\mathfrak{t}_{\mathbb{C}}$ is a one-dimensional complex vector space, let H be a basis vector. First assume $G \neq T$, so that Φ is nonempty. Then the spaces $\mathfrak{X}_\alpha$ are just eigenspaces of H on $\mathfrak{p}_{\mathbb{C}}$. Since T is one-dimensional, so is $V = \mathbb{R} \otimes_{\mathbb{Z}} X^*(T)$. Ergo if $\alpha \in \Phi$, every $\beta \in \Phi$ is of the form $s\alpha$ for some nonzero constant s. Choose α st all $|s| \geq 1$ (i.e., choose the shortest root). Let $0 \neq X_\alpha \in \mathfrak{X}_\alpha$, and let $X_{-\alpha} = -c(X_\alpha) \in \mathfrak{X}_{-\alpha}$.

Consider the complex vector space

$$W = \mathbb{C} X_{-\alpha} \oplus \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\substack{s\alpha \in \Phi \\ s>0}} \mathfrak{X}_{s\alpha}$$

By the preceding proposition, each component is mapped into another by $ad(X_\alpha)$ and $ad(X_{-\alpha})$: For example, $ad(X_{-\alpha})$ kills $X_{-\alpha}$, shifts $\mathfrak{t}_{\mathbb{C}}$ to $\mathbb{C}X_{-\alpha}$, and shifts $\mathfrak{X}_{\lambda\alpha}$ into $\mathfrak{t}_{\mathbb{C}}$ if $\lambda=1$ or $\mathfrak{X}_{(\lambda-1)\alpha}$ if $\lambda \neq 1$, and $\lambda - 1 > 0$, so this last term in in W. Moreover, $[X_\alpha, X_{-\alpha}] \in \mathfrak{t}_{\mathbb{C}} \Rightarrow [X_\alpha, X_{-\alpha}]$ is a nonzero linear multiple of H. But the commutator of two linear transformations on a finite-dim'l vector space has trace zero, so the trace of H on W is zero.

However, if $C = \overbrace{d\alpha(H)}^{\text{complex number}}$, the trace of $ad(H)$ on $\mathfrak{X}_{\lambda\alpha}$ is $\lambda \cdot C \cdot \dim(\mathfrak{X}_{\lambda\alpha})$, and the trace of $ad(H)$ on $\mathbb{C}X_{-\alpha}$ is $-C$, and the trace of $ad(H)$ on $\mathfrak{t}\mathbb{C}$ is 0. So the total trace of H on W is $-C + \sum_{\lambda \geq 1} \lambda C \dim(\mathfrak{X}_{\lambda\alpha}) = 0$. We see there is exactly one $\mathfrak{X}_{\lambda\alpha}$ w/ $\lambda > 0$, namely $\mathfrak{X}_\alpha$, and $\dim(X_\alpha) = 1$. So $\mathfrak{g} = \mathbb{C}H \oplus \mathbb{C}X_\alpha \oplus \mathbb{C}X_{-\alpha}$ is 3-dimensional. $\square$

Now we turn our attention to the general case, If $\alpha \in \Phi$, let $T_\alpha \subseteq T$ be the kernel of α. This is a closed (not necessarily connected!) subgroup w/ Lie algebra $\mathfrak{t}_\alpha$ the kernel of $d\alpha$.

Propn (i) If $\alpha \in \Phi$, $\dim(\mathfrak{X}_\alpha) = 1$

(ii) If $\alpha, \beta \in \Phi$ and $\alpha = \lambda\beta$, $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$.

Pf Let $H = C_G(T_\alpha)$ be the centralizer of T_α, in which T_α is a normal subgroup. The Lie algebra of H is the centralizer $\mathfrak{h}$ of $\mathfrak{t}_\alpha$ in $\mathfrak{g}$, so $\mathfrak{h}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\substack{\lambda\alpha \in \Phi \\ \lambda \in \mathbb{R}}} \mathfrak{X}_{\lambda\alpha}$. [Since certainly $\mathfrak{t}_\alpha$ commutes w/ every element in $\mathfrak{t}_\mathbb{C}$, and $[H, X_\alpha] = d\alpha(H) X_\alpha = 0$ for $H \in \mathfrak{t}_\alpha$, similarly for $[H, X_{\lambda\alpha}]$, but if $[H, X_\beta] = d_\beta(H) X_\beta$ for all $H \in \mathfrak{t}_\alpha$, then $d\beta$ has the same kernel as $d\alpha$ $\Rightarrow$ β is a scalar multiple of α. Ergo the maximal torus of the group H/T_α is T/T_α, which is one-dimensional. By the preceding proposition, its complexified Lie algebra is three-dimensional. But $\bigoplus \mathfrak{X}_{\lambda\alpha}$ is embedded injectively in this Lie algebra, so $\lambda = \pm 1$ are the only values of λ, and the spaces $\mathfrak{X}_{\pm\alpha}$ are one-dim'l.

Wednesday Proving that the roots always form a reduced root system.

Monday Classification of root systems & pictures (no proofs)