Lecture 25: Root Systems (Part I)

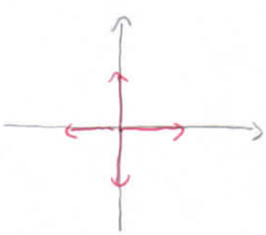
Let V be a Euclidean vector space w/ inner product $\langle , \rangle$. Given $\alpha \neq 0$ in V, let $s_\alpha(x) = x - \dfrac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha \rangle}\alpha$ be the reflection attached to α. (Geometrically, the reflection in the plane perpendicular to α.) Note that $s_\alpha(\alpha) = -\alpha$, $s_\alpha(x) = x$ if $\langle x, \alpha\rangle = 0$.

<u>Defn</u> Let V be a finite-dim'l real Euclidean space, $\Phi \subseteq V$ a finite subset of nonzero vectors. Then Φ is a <u>root system</u> if

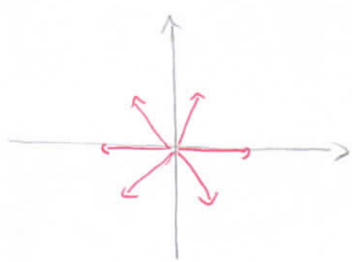
- For all $\alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$
- For all $\alpha, \beta \in \Phi$, $\dfrac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}$

We say that the root system is <u>reduced</u> if $\alpha, \lambda\alpha \in \Phi$, $\lambda \in \mathbb{R}$ implies that $\lambda = \pm 1$.

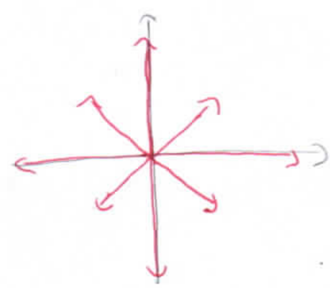
<u>Examples in $\mathbb{R}^2$</u>



$A_1 \times A_1$



A_2



C_2

Associated concept: the root datum

A <u>root datum</u> is a quadruple $(\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$ as follows.

- Λ is a lattice (a free $\mathbb{Z}$-module).
- $\Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$ is the dual lattice.
- $\Phi \subseteq \Lambda$ and $\Phi^\vee \subseteq \Lambda^\vee$ are finite sets of nonzero vectors st
  - There is a bijection $\alpha \mapsto \alpha^\vee$ from Φ to $\Phi^\vee$
  - $\alpha^\vee(\alpha) = 2$ and $\alpha^\vee(\Phi) \subseteq \mathbb{Z}$

Note that using this quadruple we may define linear maps $s_\alpha : \Lambda \to \Lambda$ and $s_{\alpha^\vee} : \Lambda^\vee \to \Lambda^\vee$ of order 2 by the formulas

$$s_\alpha(v) = v - \alpha^\vee(v)\alpha \qquad s_{\alpha^\vee}(v^*) = v^* - v^*(\alpha)\alpha^\vee$$

for each α. Easy to check these maps are adjoints, i.e.

$$s_{\alpha^\vee}(v^*)(v) = v^*(s_\alpha^{-1} v).$$

<u>Relationship</u> Our root systems will come to us w/ a lattice Λ which spans V such that $\Phi \subset \Lambda$ and Λ is invt under the maps s_α. The dual space V^* w/ dual lattice $\Lambda^\vee$ equal to the set of linear functionals $v^* : V \to \mathbb{R}$ whose image lies in $\mathbb{Z}$ is then easily constructed, and we get maps $\alpha^\vee(x) = \dfrac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}$.

If α is a root, $\alpha^\vee$ is the associated coroot. Then $(\Lambda, \Phi, \Lambda^\vee, \Phi^\vee)$ is a root datum.

<u>Classical goal</u> Associate a (reduced) root system to any cpt ctd Lie group G.

<u>Slightly better goal</u> Associate the entire root datum to G.

<u>Idea</u> Let G be a cpt ctd Lie group, T a maximal torus in G. Let $\dim(T) = r$ be the <u>rank</u> of G and $\dim(T) - \dim(Z(G))$ be the <u>semisimple rank</u>.

- Lattice $\Lambda = X^*(T)$ the group of characters of T
- Vector space $V = \mathbb{R} \otimes \Lambda$

Elements of Λ are called <u>weights</u>, Λ is the <u>weight lattice</u>. The Weyl group $W = N(T)/T$ acts on T by conjugation, hence on V. We give V an inner product which is W-invt. (Always possible for a finite group acting on a real vector space.) Now if $\rho: G \to GL(V)$ is a complex representation, we may restrict ρ to T, and $\rho|_T$ decomposes into one-dimensional characters. The elements of $\Lambda = X^*(T)$ which occur in $\rho|_T$ are the weights of the repres[illegible]n. A <u>root of G with respect to T</u> is a nonzero weight of the adjoint representation.

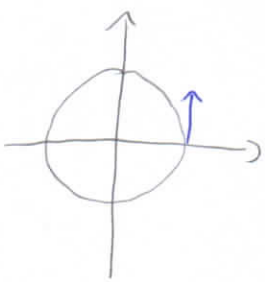
<u>Another conceptualization</u> Recall $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ where $\mathfrak{p}$ is a direct sum of 2-dim'l real irreps of the torus. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$, and $\mathfrak{p}_{\mathbb{C}}$ must decompose as pairs of one-dim'l cpx subspaces

corresponding to a root α and its opposite $-\alpha$. Hence a root is a character of T occurring in the adjoint rep on $\mathfrak{p}_{\mathbb{C}}$.

Let $\mathfrak{X}_\alpha \subseteq \mathfrak{p}_{\mathbb{C}}$ be the α-eigenspace.

Let $\Phi \subseteq V$ be the set of roots of G wrt T. The claim is that Φ is a root system.

<u>Remark</u> There are actually two ways to embed $\Lambda = X^*(T) \simeq \mathbb{Z}^r$ into a real vector space. One is to straightforwardly take the tensor product. Alternately, we note that a character $\lambda : T \to S^1$ has $d\lambda : \mathfrak{t} \to i\mathbb{R}$, where $S^1 \subseteq \mathbb{C}^\times$ in the usual way. Extending



$d\lambda$ to a complex linear map we have $d\lambda : i\mathfrak{t} \to \mathbb{R}$. The construction will produce elements $H_\alpha \in i\mathfrak{t}$ st for $\lambda \in X^*(T)$, we have $d\lambda(H_\alpha) = \alpha^\vee(\lambda)$. Both H_α and $\alpha^\vee$ are called <u>coroots</u> of α.

Before the general case, a concrete example.

<u>Example 1</u> $SU(2)$

- $\mathfrak{g} = \mathfrak{su}(2) = \{\text{skew-Hermitian matrices of trace } 0\}$
- $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$

Let $T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in S^1 \right\}$ be the group of diagonal matrices

in $SU(2)$. The characters of T are λ_k st $\lambda_k\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = t^k$, the characters of S^1.

Let $H_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $X_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

Claim $H_\alpha \in i\mathfrak{t}$ is the coroot, and X_α and $X_{-\alpha}$ span the 1-dim'l weight spaces $\mathfrak{X}_\alpha$ and $\mathfrak{X}_{-\alpha}$. The root system is then $\{\alpha, -\alpha\}$.

We say that λ_k is the highest weight in an irreducible representation V of G if k is maximal st λ_k occurs in the restriction of the representation to T.

Propn If $k \in \mathbb{Z}$ then $d\lambda_k(H_\alpha) = k$. The roots of $SU(2)$ are $\alpha = \lambda_2$ and $-\alpha = \lambda_{-2}$. If k is a nonnegative integer then $SU(2)$ has a unique irrep w/ highest weight λ_k. The weights of this representation are λ_ℓ w/ $-k \le \ell \le k$ and $\ell \equiv k \pmod 2$.

Pf We note that although H_α does not lie in $\mathfrak{t} \cong i\mathbb{R}$, iH_α does and

$$d\lambda_k(iH_\alpha) = \frac{d}{dt} \lambda_k \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \Bigg|_{t=0}$$

$$= \frac{d}{dt} e^{ikt} \Big|_{t=0}$$

$$= ik$$

So we conclude that $d\lambda_k(H_\alpha) = k$. We have

$$\mathrm{Ad}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}(X_\alpha) = \frac{d}{ds}\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right)\Bigg|_{s=0}$$

$$= \frac{d}{ds}\begin{pmatrix} 1 & t^2 s \\ 0 & 1 \end{pmatrix}\Bigg|_{s=0}$$

$$= \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix} = t^2 X_\alpha$$

So X_α spans a T-invt subspace on which T acts by the character λ_2, which is therefore a root α of $SU(2)$ wrt T.

The rest follows from the irreps of $\mathfrak{sl}(2,\mathbb{C})$.

Rephrase for clarity

$$\mathfrak{su}(2) = \langle iH_\alpha, X_\alpha, X_{-\alpha}\rangle = \mathfrak{t} \oplus \mathfrak{p}$$

X_α and $X_{-\alpha}$ generate the irreducible subspaces of the adjoint rep restricted to $\mathfrak{p}$ which have nonzero character. Their characters are the weights $\{\alpha, -\alpha\} = \{\lambda_2, \lambda_{-2}\}$

Root system of rank one.

Example 2 $G = Sp(4)$ is a maximal compact subgroup of $Sp(4,\mathbb{C})$.

$$Sp(4,\mathbb{C}) = \{ g \in GL(4,\mathbb{C}) : g J {}^t g = J \}, \quad J = \begin{pmatrix} 0 & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & 0 & \end{pmatrix}$$

Remark This is conjugate to the more usual description.

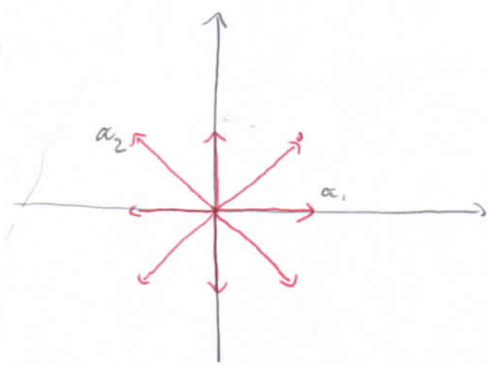
Then $Sp(4) = Sp(4,\mathbb{C}) \cap U(4)$.

A maximal torus T can be taken to be the diagonal elements. We claim that the roots are the set of eight characters

$$T \ni t = \begin{pmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & t_2^{-1} & \\ 0 & & & t_1^{-1} \end{pmatrix} \longmapsto \begin{cases} \alpha_1(t) = t_1 t_2^{-1} \\ \alpha_2(t) = t_2^2 \\ \alpha_1+\alpha_2(t) = t_1 t_2 \\ 2\alpha_1+\alpha_2(t) = t_1^2 \\ -\alpha_1(t) = t_1^{-1} t_2 \\ -\alpha_2(t) = t_2^{-2} \\ -(\alpha_1+\alpha_2)(t) = t_1^{-1} t_2^{-1} \\ -(2\alpha_1+\alpha_2)(t) = t_1^{-2} \end{cases}$$

$t_1, t_2 \in S^1$

i.e



Exercise This is in fact a root system.

How to start thinking about this?

The complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ consists of matrices of the form

$$\begin{pmatrix} t & x_{12} & x_{13} & x_{14} \\ x_{21} & t_2 & x_{23} & x_{13} \\ x_{31} & x_{32} & -t_2 & -x_{12} \\ x_{41} & x_{31} & -x_{21} & -t_1 \end{pmatrix}$$

The spaces $\mathfrak{X}_{\alpha_1}$ and $\mathfrak{X}_{-\alpha_1}$ are spanned by the vectors

$$X_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad X_{-\alpha_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$X_{\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad X_{-\alpha_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

<u>Exercise</u> $Ad(t)X_\alpha = \alpha(t)X_\alpha$ for each of α_1 and α_2, proving that these characters are roots. Similarly for the others.

<u>Remark</u> The matrices $X_{\alpha_1}, X_{\alpha_2}$ really do live in the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, not in $\mathfrak{g}$, since skew-Hermitian matrices have $x_{ij} = -\overline{x_{ji}}$ and the t_i purely imaginary.

<u>Remark</u> In this example

$$H_{\alpha_1} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \qquad H_{\alpha_2} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix} \qquad \text{etc}$$

In each case $-iH_\alpha \in \mathfrak{t}$.

The H_α satisfy $[H_\alpha, X_\alpha] = 2X_\alpha$, $[H_\alpha, X_{-\alpha}] = -2X_{-\alpha}$.

<u>Next Time</u> The general construction of the root system.