Lecture 24: The Weyl Integration Formula

Ref Bump Chapter 17

- G a compact Lie group, T a maximal torus.
- Last Time T meets every conjugacy class, so it should be possible to integrate a class function over G w/ reference only to T.

To understand this we want the Riesz representation thm from measure theory: Let X be a locally cpt Hausdorff space and $C_c(X)$ the set of cts compactly-supported functions on X. A linear functional I on $C_c(X)$ is called positive if $I(f) \geq 0$ for every nonnegative f. Then each such I is of the form $I(f) = \int_X f \, d\mu$ for some regular Borel measure $d\mu$.

With this in mind...

Propn Let G be a locally cpt group, and H a compact subgroup. Let $d\mu_G$ and $d\mu_H$ be left Haar measures on G and H. Then $\exists$ a regular Borel measure $d\mu_{G/H}$ on G/H invariant under the action of G by left translation. The measure $d\mu_{G/H}$ may be normalized such that, for $f \in C_c(G)$, we have

$$\int_{G/H} \underbrace{\int_H f(gh) \, d\mu_H(h)}_{\substack{\text{Constant on cosets } gH, \\ \text{hence identified w/} \\ \text{a function on } G/H}} \, d\mu_{G/H}(gH)$$

PF We can choose $d\mu_H$ so H has total volume 1. We define a map $\lambda : C_c(G) \longrightarrow C_c(G/H)$ via $(\lambda F)(g) = \int_H F(gh)\, d\mu_H(h)$. Here we regard λF as a function on G/H since it is invariant under right-translation by elements of H. Since H is compact, λF is automatically compactly supported. Moreover, if $\phi \in C_c(G/H)$, then treating ϕ as a function on G, we have $\lambda\phi = \phi$ since

$$\lambda\phi(g) = \int_H \phi(gh)\, d\mu_H(h) = \int_H \phi(g)\, d\mu_H(h) = \phi(g)$$

ϕ is invt on cosets

So λ is surjective. Define a linear functional I on $C_c(G/H)$ via $I(\lambda F) = \int_G F(g)\, d\mu_G(g)$ for $F \in C_c(G)$. We need to check this is well-defined, i.e. that if $\lambda F = 0$ then $\int_G F(g)\, d\mu_G(g) = 0$. To see this, note that the function $(g,h) \mapsto F(gh)$ is compactly supported and cts on $G \times H$, so if $\lambda F = 0$, Fubini's Thm says that

$$\begin{aligned} 0 &= \int_G (\lambda F)(g)\, d\mu_G(g) \\ &\quad\Big\} \text{ Defn} \\ &= \int_G \int_H F(gh)\, d\mu_H(h)\, d\mu_G(g) \\ &\quad\Big\} \text{ Fubini} \\ &= \int_H \int_G F(gh)\, d\mu_G(g)\, d\mu_H(h) \end{aligned}$$

Now make the variable change $g \mapsto gh^{-1}$. Since $d\mu_H$ is a <u>left</u> Haar measure, in principle the integral might not be invt under this change, but since H is compact, by the usual arguments $d\mu_H$ is also a right Haar measures, so the last line becomes

$$0 = \int_H \int_G F(g)\, d\mu_G(g)\, d\mu_H(h) = \int_G F(g)\, d\mu_G(g) \quad \checkmark.$$

Therefore I is well-defined. Now the existence of the measure on G/H follows from the Riesz representation thm. $\square$

__Exercise__ For the case of a cpt Lie group w/ a closed Lie subgroup, derive this result from differential geometry.

__Recall__ If G is a cpt Lie group w/ maximal torus T, then the Lie algebra $\mathfrak{t} \leq \mathfrak{g}$ is an invt subspace, w/ orthogonal complement a subspace $\mathfrak{p}$ that decomposes as the direct sum of nontrivial two-dim'l real irreps of T.

Let $W = N(T)/T$ be the Weyl group of G. W acts on T by conjugation. If $w = nT$ a coset is an element of W, then given $t \in T$, the element ntn^{-1} actually only depends on w, so we write it wtw^{-1}.

__Thm__ (i) Two elements of T are conjugate in $G \iff$ they are conjugate in $N(T)$.

(ii) The inclusion $T \longrightarrow G$ induces a bijection between the orbits of W on T and the conjugacy classes of G.

__Pf__ Suppose that $t, u \in T$ are conjugate in G, say $gtg^{-1} = u$. Let $H = C_G(u)^\circ$ be the connected component of the identity in the centralizer of u. Last time we saw that this is a closed Lie subgroup. Now T and gTg^{-1} are both connected commutative groups containing u, and therefore they are both contained in H. Since they are both maximal tori in G, they are also maximal

tori in H, hence they are conjugate in the compact connected group H. Pick $h \in H$ st $hTh^{-1} = gTg^{-1}$, and then $w = h^{-1}g \in N(T)$.
Since $wtw^{-1} = h^{-1}uh = u$, we see that t and u are conjugate in $N(T)$.

For (iii), note that G is the union of the conjugates of T, so (iii) follows from (i).

<u>Propn</u> The centralizer $C(T)$ of T is exactly T.

<u>Pf</u> Since $C(T) \subseteq N(T)$, certainly T is of finite index in $C(T)$. So if $x \in C(T)$, $x^n \in T$ for some n. Let t_0 be a generator of T. Since the nth power map $T \longrightarrow T$ is surjective, $\exists$ $t \in T$ st $(xt)^n = t_0$. Now xt is contained in a maximal torus T' which contains $t_0 \Rightarrow T \subset T'$. By maximality, $T' = T$ and $x \in T$. □

<u>Propn</u> There is a dense open set $\Omega \in T$ st the $|W|$ elements wtw^{-1} for $w \in W$ are all distinct for each $t \in \Omega$.

<u>Pf</u> If $w \in W$, let $\Omega_w = \{t \in T : wtw^{-1} \neq t\}$. This is an open subset of T (since its complement is certainly closed!). If $w \neq 1$ and t is a generator of T, then $t \in \Omega_w$ because otherwise if $n \in N(T)$ represents w, then $n \in C(t) = C(T) = T$, which is a contradiction since we assumed that $w \neq 1$. The set $\Omega = \bigcap_{w \neq 1} \Omega_w$ is the dense in T since it contains all generators of T and the set of generators is itself dense in T. □

Thm (Weyl) Let G be a cpt connected Lie group, T a maximal torus, $\mathfrak{p}$ a vector space complement to $\mathfrak{t}$ in $\mathfrak{g}$. If F is a class function and dg and dt are Haar measures on G and T normalized st G and T have volume 1, then

$$\int_G F(g)\,dg = \frac{1}{|W|}\int_T F(t)\det\left(\left[\mathrm{Ad}(t^{-1}) - Id_{\mathfrak{p}}\right]\Big|_{\mathfrak{p}}\right)dt$$

Pf Let $X = G/T$, and give X the measure dx invt under left translation by G st X has volume 1. Consider the map

$$\begin{aligned} \phi : X \times T &\longrightarrow G \\ (xT, t) &\longmapsto xtx^{-1} \end{aligned}$$

Choose volume elements on $\mathfrak{g}$ and $\mathfrak{t}$ so that the Jacobians of the exponential maps $\mathfrak{t} \to T$ and $\mathfrak{g} \to G$ (which each have the identity as their derivative) are 1.

Let us compute the Jacobian $J\phi$ of ϕ. Parametrize a nbhd of xT in X using a chart from a neighborhood of the origin in $\mathfrak{p}$, i.e. via the map $U \mapsto x\exp(U)T$ for U in some nbhd of the origin in $\mathfrak{p}$. We also use the exponential map to parametrize a neighborhood of $t \in T$, using the map $V \mapsto t\exp(V)$ for $V \in \mathfrak{t}$. So near $(xT, t) \in X \times T$ we have a chart given by $(U, V) \mapsto (x\exp(U), t\exp(V))$ $\in X \times T$

from some neighborhood of the origin in $\mathfrak{p} \times \mathfrak{t}$. In these coordinates, ϕ is given by $(U, V) \mapsto x\exp(U)\,t\exp(V)\exp(-U)x^{-1} \in G$.

We translate on the left by $t^{-1}x^{-1}$ and on the right by x, which does not affect the Jacobean since the Haar measure is translation is translation invt. This leaves us with the map

$$(U,V) \mapsto t^{-1}\exp(U)t\exp(V)\exp(-U) = \exp(Ad(t^{-1})U)\exp(V)\exp(-U)$$

We identify the tangent space of $p \times t$ with itself, i.e. w/ $g = p \oplus t$, and then the differential of this map at 0 is

$$U + V \longmapsto Ad(t^{-1})U + V - U = (Ad(t^{-1}) - I_p)U + V$$

The Jacobian is the determinant of the differential, so

$$(J\varphi)(xT, t) = \det([Ad(t^{-1}) - I_p]|_p)$$

Now by the last proposition, the map $\varphi : X \times T \longrightarrow G$ is a $|W|$-fold cover over a dense open set, so for any class function F on G, we have

$$\int_G f(g)dg = \frac{1}{|W|}\int_{X \times T} f(\varphi(xT, t))J(\varphi(xT, t))dx \times dt$$

Now $F(\varphi(xT, t))J(\varphi(xT, t)) = F(t)\det([Ad(t^{-1}) - I_p]|_p)$ is independent of x since f is a class fcn. This implies the result.

Example Let $G = U(n)$ and T be the diagonal torus. Writing

$$t = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} \in T$$

w/ $\int_T dt = 1$ the Haar measure, we have

$$\int_G F(g)dg = \frac{1}{n!}\int_T F\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \prod_{i<j} |t_i - t_j|^2 dt$$

For F any class function.

PF We need to check that $\det([Ad(t^{-1}) - I_p]|_p) = \prod_{i<j} |t_i - t_j|^2$.
To compute this determinant we may as well consider the induced transformation on $\mathbb{C}\otimes p$. Now $\mathbb{C}\otimes u(n) \simeq gl(n, \mathbb{C}) = Mat_n(\mathbb{C})$. Moreover $\mathbb{C}\otimes p$ is spanned by the T-eigenspaces in $\mathbb{C}\otimes u(n)$ corresponding to nontrivial characters of T. These are spanned by elementary matrices E_{ij}; the eigenvalue of $Ad(t^{-1})$ on E_{ij} is $t_i t_j^{-1}$. Ergo

$$\det([Ad(t^{-1}) - I_p]|_p) = \prod_{i \neq j} (t_i t_j^{-1} - 1) = \prod_{i<j} (t_i t_j^{-1} - 1)(t_j t_i^{-1} - 1)$$

As $|t_i| = |t_j| = 1$, we have $(t_i t_j^{-1} - 1)(t_j t_i^{-1} - 1) = (t_i - t_j)(t_i^{-1} - t_j^{-1}) = |t_i - t_j|^2$.

$\square$