Ref Bump Chapter 17

- $G$ a compact Lie group, $T$ a maximal torus.

- Last time $T$ meets every conjugacy class, so it should be possible to integrate a class function over $G$ w/ reference only to $T$.

To understand this we want the Riesz representation theorem from measure theory: Let $X$ be a locally compact Hausdorff space and $C_c(X)$ the set of cts compactly supported functions on $X$. A linear functional $I$ on $C_c(X)$ is called positive if $I(f) \geq 0$ for every nonnegative $f$.

Then each such $I$ is of the form $I(f) = \int_X f \, d\mu$ for some regular Borel measure $\mu$.

With this in mind...

Prop. Let $G$ be a locally compact group, and $H$ a compact subgroup. Let $dG$ and $dH$ be left Haar measures on $G$ and $H$.

Then if a regular Borel measure $dG/H$ on $G/H$ is invariant under the action of $G$ by left translation. The measure $dG/H$ may be normalized such that, for $f \in C_c(G)$, we have

$$\int_{G/H} \int_H f(gh) \, d\mu_H(h) \, d\mu_G(\gamma H)$$

is constant on cosets $gH$, hence identified w/ a function on $G/H$. 
We can choose $\delta_H$ so $H$ has total volume 1. We define a map $A: \mathcal{C}_c(G) \to \mathcal{C}_c(G/H)$ via $(AF)(g) = \int_H f(gh) \, d\mu_H(h)$. Here we regard $A$ as a function on $G/H$ since it is invariant under right-translation by elements of $H$. Since $H$ is compact, $A F$ is automatically compactly supported. Moreover, if $\phi \in \mathcal{C}_c(G/H)$, then treating $\phi$ as a function on $G$, we have $A F = \phi$ since

$$
A F(g) = \int_H f(gh) \, d\mu_H(h) = \int_H \phi(g) \, d\mu_H(h) = \phi(g)
$$

is not on cosets.

So $A$ is surjective. Define a linear functional $\mathcal{L}$ on $\mathcal{C}_c(G/H)$ via $\mathcal{L}(AF) = \int_G F(g) \, d\mu_G(g)$ for $F \in \mathcal{C}_c(G)$. We need to check this is well-defined, i.e. that if $AF = 0$ then $\int_G F(g) \, d\mu_G(g) = 0$.

To see this, note that the function $(g, h) \mapsto f(gh)$ is compactly supported and acts on $G \times H$, so if $AF = 0$, Fubini's Theorem says that

$$
0 = \int_G (AF)(g) \, d\mu_G(g)
$$

$$
= \int_G \int_H f(gh) \, d\mu_H(h) \, d\mu_G(g)
$$

$$
= \int_H \int_G F(g) \, d\mu_G(g) \, d\mu_H(h)
$$

Now make the variable change $g \mapsto gh$. Since $d\mu_H$ is a left Haar measure, in principle the integral might not be in $G$, under this change, but since $H$ is compact, by the usual arguments $d\mu_H$ is also a right Haar measure, so the last line becomes

$$
0 = \int_H \int_G F(g) \, d\mu_G(g) \, d\mu_H(h) = \int_G F(g) \, d\mu_G(g)
$$

$\checkmark$. 

Therefore I is well-defined. Now the existence of the measure
on $G/H$ follows from the Riesz representation thm. $\square$

**Exercise** For the case of a cpt lie group w/ a closed
lie subgroup, derive this result from differential geometry.

**Recall** If $G$ is a cpt lie group w/ maximal torus $T$, then the lie
algebra $\mathfrak{g}$ is an invt subspace, w/ orthogonal complement a
subspace $\mathfrak{t}$ that decomposes as the direct sum of non-trivial
two-dim'l real reps of $T$.

Let $W = \mathcal{N}(T)/T$ be the Weyl group of $G$. $W$ acts on $T$ by
conjugation. If $w \in T$ a coset is an element of $W$, then given
$e_T$, the element $nt^{-1}$ actually only depends on $w$, so we
write it $wtw^{-1}$.

**Thm** (1) Two elements of $T$ are conjugate in $G$ (2) they are
conjugate in $N(T)$.

(2) The inclusion $T \rightarrow G$ induces a bijection between the orbits
of $W$ on $T$ and the conjugacy classes of $G$.

**Pf** Suppose that $g, u \in T$ are conjugate in $G$, say $g^{-1}ug \in T$. Let $H$
be the connected component of the identity in the
centralizer of $u$. Last time we saw that this is a closed
lie subgroup. Now $T$ and $g^{-1}ug^{-1}$ are both connected commutative
groups containing $u$, and therefore they are both contained in $H$.
Since they are both maximal tori in $G$, they are also maximal
tori in $H$, hence they are conjugate in the compact connected group $H$. Pick $h \in H$ s.t $hTh^{-1} = gTg^{-1}$, and then $w = hg \in N(T)$.

Since $wT^{-1} = h^{-1}gh = g$, we see that $t$ and $u$ are conjugate in $N(T)$.

For (ii), note that $G$ is the union of the conjugates of $t$, so (ii) follows from (i).

**Prop.** The centralizer $C(T)$ of $T$ is exactly $T$.

**Pr.** Since $C(T) \subseteq N(T)$, certainly $T$ is of finite index in $C(T)$.

So if $x \in C(T)$, $x \in T$. For some $n$, let $t$ to be a generator of $T$.

Since the nth power map $T \to T$ is surjective, $\exists t \in T$ s.t $\exists x \in T$ s.t $\exists t^n = x_0$. Now $x_0$ is contained in a maximal torus $T'$ which contains $t_0 \to T \in T$. By maximality, $T' = T$ and $x \in T$. $\square$

**Prop.** There is a dense open set $\Omega \subseteq T$ s.t the $w \in T$ elements $w^{-1}$

For $w \in \Omega$ are all distinct for each $w \in \Omega$.

**Pr.** If $w \in \Omega$, let $\Omega_w = \{ x \in T : w^{-1}xw \in T \}$. This is an open subset of $T$ (since its complement is certainly closed). If $w \in T$ and $t$ is a generator of $T$, then $t \in \Omega_w$ because otherwise if $w \in N(T)$ represents $w$, then $w \in C(T) = C(T) = T$, which is a contradiction since we assumed that $w \notin T$. The set $\Omega = \bigcap \Omega_w$ is the dense subset in $T$ since $t$ contains all generators of $T$ and the set of generators is itself dense in $T$. $\square$
Thm (way1) Let $G$ be a cpt connected Lie group, $T$ a maximal torus, $p$ a vector space complement to $x$ in $gr$. If $F$ is a class function and $dg$ and $dx$ are Haar measures on $G$ and $T$ normalized so $G$ and $T$ have volume $1$, then

$$\int_G f(x) \, dg = \frac{1}{|w|} \int_T F(x) \det \left( \left[ \text{Ad}(e^y) - \text{Id}_g \right] p \right) \, dx$$

Proof. Let $x = G/T$, and give $X$ the measure $dx$ invariant under left translation by $G$ so $X$ has volume $1$. Consider the map

$$\Phi : x \times T \to G$$

$$(xT, t) \mapsto xtx^{-1}$$

Choose volume elements on $G$ and $T$ so that the Jacobians of the exponential maps $x \to T$ and $y \to G$ (which each have the identity as their derivative) are $1$.

Let us compute the Jacobian $J_\Phi$ of $\Phi$. Parametrize a nbhd of $xT$ in $X$ using a chart from a neighborhood of the origin in $p$, i.e., via the map $V \to x \exp(xT)$ for $V$ in some nbhd of the origin in $p$. We also use the exponential map to parametrize a neighborhood of $T \times T$, using the map $V \to \exp(x)$ for $V$ in $T$. So near $(xT, t) \in X \times T$ we have a chart given by $(y, v) \mapsto (x \exp(y) \exp(v)) \in X \times T$.

From some neighborhood of the origin in $p \times T$. In these coordinates, $\Phi$ is given by $(y, v) \mapsto x \exp(v) \exp(y) \exp(-v) x^{-1} \in G$.

We translate on the left by $x^{-1}$ and on the right by $x$, which does not affect the Jacobian since the Haar measure is translation.
is translation inv. This leaves us with the map

\[(x, y) \mapsto \exp(x) \cdot \exp(y) \exp(-x) = \exp(\operatorname{Ad}(e^y) x) \exp(y) \exp(-x)\]

We identify the tangent space of \(pt\) with itself, i.e. \(w/\mathfrak{g} = pt\),
and then the differential of this map at \(0\) is

\[U + V \mapsto \operatorname{Ad}(e^y) U + V - U = (\operatorname{Ad}(e^y) - I_\mathfrak{g}) U + V\]

The Jacobian is the determinant of the differential, so

\[
\begin{align*}
\det \left( \left[ \operatorname{Ad}(e^y) - I_{\mathfrak{g}} \right] \big|_{\mathfrak{g}} \right)
\end{align*}
\]

Now by the last proposition, the map \(\varphi: \mathbb{R} \times \mathbb{R} \to G\) is a \(\mathbb{R}\)-fied cover over a dense open set, so for any class function \(F\) on \(G\),
we have

\[
\int_G F(g) dg = \frac{1}{\mathbb{R}} \int_{\mathbb{R} \times \mathbb{R}} F(y(x, \xi)) \varphi((x, \xi)) \, dx \, d\xi
\]

Now \(F(y(x, \xi)) \varphi((x, \xi)) = F(x) \cdot \det \left( \left[ \operatorname{Ad}(e^y) - I_\mathfrak{g} \right] \big|_{\mathfrak{g}} \right)\) is independent of \(x\) since \(F\) is a class function. This implies the result.
Example. Let $G = U(n)$ and $T$ be the diagonal torus. Writing

$$t = \begin{pmatrix} e_1 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_n \end{pmatrix} \in T$$

w.l.o.g. $t_{ij} = 1$ be the Haar measure, we have

$$\int_{G} F(g) dg = \frac{1}{n!} \int_{T} F \left( \begin{pmatrix} e_1 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e_n \end{pmatrix} \right) \prod_{i<j} |t_{ij} - t_{ij}^{-1}|^2 dt$$

For $F$ any class function.

**Proof.** We need to check that $\det \left( \text{Ad}(t^{-1}) - I_p \right) = \prod_{i<j} |t_{ij} - t_{ij}^{-1}|^2$.

To compute this determinant, we may as well consider the induced transformation on $O(p)$. Now $O(n) \cong U(n) \cong \text{Mat}_n(\mathbb{C})$. Moreover, $O(p)$ is spanned by the $T$-eigenspaces in $\text{Mat}_n(\mathbb{C})$ corresponding to nontrivial characters of $T$. These are spanned by elementary matrices $E_{ij}$; the eigenvalue of $\text{Ad}(t)$ on $E_{ij}$ is $t_{ij} - t_{ij}^{-1}$. Ergo

$$\det \left( \text{Ad}(t^{-1}) - I_p \right) = \prod_{i<j} (t_{ij} - t_{ij}^{-1}) = \prod_{i<j} (t_{ij}^{-1} - t_{ij}) (t_{ij} - t_{ij}^{-1})$$

As $1_{ij} = 1_{ij}^{-1} = 1$, we have $(t_{ij}^{-1} - t_{ij}) (t_{ij} - t_{ij}^{-1}) = (t_{ij} - t_{ij}^{-1}) (t_{ij}^{-1} - t_{ij}) = t_{ij} - t_{ij}^{-1}$.