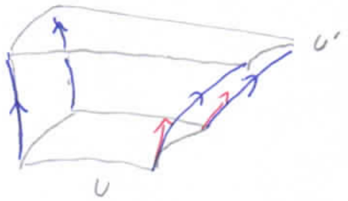
Lecture 23: Geodesics & Maximal Tori, cont'd

<u>Recall</u> Let M be a Riemannian mfd of dimn n; U a submfd of dimn $n-1$ which is homeomorphic to a disk. We choose a unit normal in a fixed direction near U. <u>Geodesic coordinates</u> near U are $(t, x_2, \ldots, x_n)$ such that $t \mapsto (t, x_1, \ldots, x_n)$ is the geodesic through x in the direction of the unit normal.

<u>Propn</u> In geodesic coordinates, $g_{1i} = 0$ for $2 \le i \le n$ and $g_{11} = 1$.



<u>Pf</u> Recall that a geodesic should locally have $\dfrac{d^2 x_k}{dt^2} = -\{ij, k\} \dfrac{dx_i}{dt}\dfrac{dx_j}{dt}$. In our case $\dfrac{dx_i}{dt} = 0$ for $i \neq 1$,

so $\dfrac{d^2 x_k}{dt^2} = 0$ for all k. Ergo $0 = -\{11, k\} \dfrac{dx_1}{dt}\dfrac{dx_1}{dt} = -\{11, k\}$,

So each $-\{11, k\} = 0$. Given that $(g_{k\ell})$ is invertible, we see that $[11, k] = 0$ as well. So $\frac{1}{2}\left(\dfrac{\partial g_{1k}}{\partial x_1} + \dfrac{\partial g_{1k}}{\partial x_1} - \dfrac{\partial g_{11}}{\partial x_k}\right) = 0 \Rightarrow \dfrac{\partial g_{1k}}{\partial x_1} = \dfrac{1}{2}\dfrac{\partial g_{11}}{\partial x_k}$ for all k.

If we take $k=1$, $\dfrac{\partial g_{11}}{\partial x_1} = \dfrac{1}{2}\dfrac{\partial g_{11}}{\partial x_1} \Rightarrow \dfrac{\partial g_{11}}{\partial x_1} = 0$, so for fixed $x_2, \ldots, x_n$, g_{11} is a constant. When $x_1 = 0$, the initial condition of the geodesic through $(0, x_2, \ldots, x_n)$ is that it is tangent to the unit normal to the surface, so in particular its tangent vector $\dfrac{\partial}{\partial x_1}$ at this point has length one. Ergo $g_{11} = 1$ when $x_1 = 0$, hence $g_{11} = 1$ throughout the geodesic coordinate neighborhood. Now consider $2 \le k \le n$ and $\dfrac{\partial g_{1k}}{\partial x_1} = \dfrac{1}{2}\dfrac{\partial g_{11}}{\partial k} = 0$ since g_{11} is constant.

Now for $x_1 = 0$, by assumption $\dfrac{\partial}{\partial x_1}$ and $\dfrac{\partial}{\partial x_k}$ are orthogonal, so

g_{1k} vanishes when $x_1 = 0$, so it vanishes for all x_1. □

We can now check that short geodesics are paths of shortest length:

Propn ① Let $p : [0,1] \to M$ be a geodesic. Then there exists an $\varepsilon > 0$ st the restriction of p to $[0,\varepsilon]$ is the unique path of shortest length from $p(0)$ to $p(\varepsilon)$.

② Let $x \in M$. There exists a nbhd N of x st for all $y \in N$ there is a unique path of shortest distance from x to y, which is a geodesic.

Pf We choose a hypersurface U orthogonal to p at $t=0$ and construct geodesic coordinates as previously. Choose ε and θ sufficiently small that the set N of points w/ coordinates $\{x_1 \in [0,\varepsilon],\ 0 \le |x_2|, \dots, |x_n| \le \theta\}$ is contained in the interior of the geodesic coordinate neighborhood. We assume that the coordinates of $p(0)$ are $(0,\dots,0)$, so that $p(t) = (t, 0, \dots, 0)$. Then $|p| = \varepsilon$, where we let $|p|$ denote the length of the restriction of p to $[0,\varepsilon]$.

Now we check that if $q : [0,\varepsilon] \to M$ is any admissible path w/ $q(0) = p(0)$ and $q(\varepsilon) = p(\varepsilon)$, then $|q| \ge |p|$. First, suppose that $q([0,\varepsilon])$ lies entirely within the neighborhood N and the x_1-coordinate of $q(t)$ is monotonically increasing. Reparametrizing q we can arrange that $q(t)$ and $p(t)$ have the same x_1 coordinate, namely t. We then write $q(t) = (\underset{\substack{\parallel \\ x_1(t)}}{t}, x_2(t), \dots, x_n(t))$. Since $g_{1k} = g_{k1} = 0$ and $g_{11} = 1$, we see that

$$|q| = \int_0^{\varepsilon} \sqrt{\sum_{i,j} g_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt}}\, dt$$

$$= \int_0^{\varepsilon} \sqrt{1 + \sum_{2\le i,j\le n} g_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt}}\, dt$$

Since (g_{ij}) is positive definite, its minor $(g_{ij})_{2\le i,j\le n}$ is also positive definite, so in particular $\sum_{2\le i,j\le n} g_{ij} \frac{dx_i}{dt}\frac{dx_j}{dt} \ge 0$, implying that $|q| \ge \int_0^{\varepsilon} \sqrt{1}\,dt = \varepsilon = |p|$.

<u>Exercise</u> Extend this argument to paths which have nonmonotonic first coordinate, or which are long enough to go outside of the neighborhood N.

For part (ii), given a unit tangent vector $X \in T_x M$, there is a unique geodesic $p_X : [0, \varepsilon_X] \to M$ through x which is tangent to X, w/ $\varepsilon_X > 0$ chosen small enough that this is a path of shortest length. Indeed, we can choose ε_X small enough that it also works for nearby unit tangent vectors Y. (To see this, pick a local diffeomorphism of M moving X to Y & regard X as fixed while the metric varies; this produces small variations of the g_{ij}, and the ε in part (i) can be used to work for sufficiently small variation of the g_{ij}.) So for each unit vector $X \in T_x M$ there is an $\varepsilon_X > 0$ and a nbhd N_X of X in the unit ball of $T_x M$ st $p_Y : [0, \varepsilon_X] \to M$ is a path of shortest length for all $Y \in N_X$. Since the unit ball in $T_x M$ is compact, it can be covered by finitely many of the nbhds N_X. Letting ε be the minimum of the corresponding ε_X, we can take N to be the set of points connected to x by a geodesic of length $< \varepsilon$.

If M is a connected Riemannian mfd, we make M into a metric space by defining $d(x,y)$ to be the infinum of $|p|$, where p is a path from x to y.

Thm Let M be a cpt connected Riemannian mfd, and let x and y be points of M. Then there is a geodesic $p : [0,1] \to M$ w/ $p(0) = x$ and $p(1) = y$.

Remark The condition in the theorem is stronger than it needs to be; this is true for manifolds that are geodesically complete, i.e. any well-paced geoodesic can be extended to $(-\infty, \infty)$.

Pf Let $\{p_i\}$ be a sequence of well-paced paths from x to y st $|p_i| \longrightarrow d(x,y)$. Because they are well-paced, if $0 \le a < b \le 1$ we have that $d(p_i(a), p_i(b)) = (b-a)|p_i|$, and it follows that the $\{p_i\}$ are equicontinuous. Hence there is a subsequence that converges uniformly to a path p. This is clearly cts (it's not immediately obvious that it's smooth). But on each sufficiently short interval $0 \le a < b \le 1$, $p(b)$ is near enough to $p(a)$ that the unique path of shortest length between them is a geodesic. It follows that p itself is a geodesic.

Now we can consider applications to Lie groups.

Thm Let G be a cpt Lie group. There is a Riemannian metric on G which is invariant under left and right translation. In this metric, any geodesic is a (right or left) translate of a path $t \mapsto \exp(tX)$ for some $X \in \mathfrak{g}$.

PF Since G is a compact group acting by Ad on the real vector space $\mathfrak{g}$, there is an Ad(G) inner product on $\mathfrak{g}$. This gives us an inner product on T_eG, which we transfer to T_gG by the left-translation map. Right-translation in principle induces a different isomorphism $T_eG \xrightarrow{R_g^*} T_gG$, but these two maps differ by the derivative of conjugation, namely $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$, so since our inner product is invariant under Ad(g), the Riemannian metric thusly defined is inv't under both left and right translation.

To show that any geodesic is a translate of the exponential map, it suffices to check that any short segment of a geodesic is of the form $t \mapsto g \cdot \exp(tX)$. But the metric is translation-inv't, so this just asks us to check that geodesics near the identity are of the form $t \mapsto \exp(tX)$.

Case I $G \simeq \mathbb{R}^n/\mathbb{Z}^n$ is a torus, and $T_eG \simeq \mathbb{R}^n$ can be assumed to have the standard inner product. Then geodesics are just quotients of straight lines in $\mathbb{R}^n$ and therefore translates of the exponential map.

Case II For general G, let $X \in \mathfrak{g}$ and $E_X : (-\varepsilon, \varepsilon) \to G$ be the geodesic through the origin tangent to $X \in \mathfrak{g}$. For $\lambda \in \mathbb{R}$, $t \mapsto E_X(\lambda t)$ is the geodesic through the origin tangent to λX, so that $E_X(\lambda t) = E_{\lambda X}(t)$. So there is a nbhd U of the origin in $\mathfrak{g}$ and a map $E : U \to G$ st $E_X(t) = E(tX)$ for $X, tX \in U$. Wts this is the same as the exponential map.

Given $g \in G$, then translating $E(tX)$ on the left by g and the right by g^{-1} gives another geodesic, whose tangent is $\mathrm{Ad}(g)X$. Thus if $tX \in U$, $gE(tX)g^{-1} = E(t\,\mathrm{Ad}(g)X)$. For fixed $X \in \mathfrak{g}$, let T be a maximal torus containing the one-parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\}$. Then $E(tX)$ commutes w/ $g \in T$ when $tX \in U$. Ergo the path $t \mapsto E(tX)$ runs through the centralizer $C(T)$ and therefore through $N(T)$. Recalling that the closed component of $N(T)$ containing the identity is T, we see that $E(tX)$ lies in T. But the geodesics in T are exactly the paths $\exp(tX)$. □

<u>Thm</u> Let G be a cpt ctd Lie group, and $\mathfrak{g}$ its Lie algebra. Then the exponential map $\mathfrak{g} \to G$ is surjective.

<u>PF</u> Put a translation-inv't Riemannian structure on G. Then given $g \in G$, there is a geodesic path from the identity to g. But this path is of the form $t \mapsto \exp(tX)$.

<u>Thm</u> Let G be a cpt ctd Lie group and T a maximal torus. There exists $k \in G$ st $g \in kTk^{-1}$.

<u>PF</u> Let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebras of G and T. Let t_0 be a generator of T. Let $X \in \mathfrak{g}$ and $H_0 \in \mathfrak{t}$ be such that $\exp(X) = g$ and $\exp(H_0) = t_0$, which is possible since exp is surjective. Since G is a compact group acting by Ad on $\mathfrak{g}$, there is an (Ad G)-inv't inner product $\langle\,,\rangle$ on $\mathfrak{g}$. Choose $k \in G$ so that the real value $\langle X, \mathrm{Ad}(k)H_0\rangle$ is maximal. (This maximum is achieved, by compactness of G.) Let $H = \mathrm{Ad}(k)H_0$. Then ⟶

$\exp(H) = kt_0k^{-1}$ is a generator for kTk^{-1}.

If $Y \in \mathfrak{g}$ is arbitrary, then $\langle X, Ad(e^{tY})H \rangle \big|_{t=0} = \langle X, ad(Y)H \rangle = \langle X, [Y,H] \rangle = -\langle X, [H,Y] \rangle$

Since the bilinear form is invariant under Ad, it is inv't under ad, meaning that $0 = \langle X, [H,Y] \rangle + \langle [H,X], Y \rangle$. So $\langle [H,X], Y \rangle = 0$ for all $Y \in \mathfrak{g}$
$\Rightarrow [H,X] = 0$. Therefore $\exp(H)$ commutes w/ $\exp(tX)$ for all $t \in \mathbb{R}$.
Since $\exp(H)$ generates kTk^{-1}, the one-parameter subgroup $\exp(tX)$ is contained in the centralizer of kTk^{-1}. But kTk^{-1} is in fact a maximal torus, so it follows that $\{\exp(tX)\} \subseteq kTk^{-1}$.
In particular, $g = \exp(X) \in kTk^{-1}$. $\square$

<u>Thm</u> (Cartan) Let G be a compact connected Lie group and T be a maximal torus. Then every maximal torus is conjugate to T, and every element of G is contained in a conjugate of T.

<u>PF</u> The second statement is implied by the previous thm. For the first statement, let T' be another maximal torus and t' a generator. Then t' is contained in kTk^{-1} for some k, so $T' \subseteq kTk^{-1}$. As both are maximal tori, they are equal.

<u>Propn</u> Let G be a cpt connected Lie group, $S \subseteq G$ a torus (not necessarily maximal), and $g \in C_G(S)$ an element of its centralizer. Let H be the closure of the group generated by S and g. Then H has a topological generator; that is, there exists $h \in H$ s.t. the subgroup generated by h is dense in H.

Pf Since H is closed and abelian, its conneted cpt H° containing the identity is a torus. Let h_0 be a generator of the torus. Now H/H° is compact and discrete, hence finite. Since $S \subseteq H^\circ$ and S and g together generate a dense subgroup of H, the finite group H/H° is cyclic & generated by gH°. Let r be the order of this group, st $g^r \in H^\circ$. Since the rth power map $H^\circ \longrightarrow H^\circ$ is surjective, $\exists\ u \in H^\circ$ st $(gu)^r = h_0$. Then the group generated by $h = ug$ contains h_0 and a generator $gH^\circ = (gu)H^\circ$ of H/H°. So h is a topological generator of H. $\square$

Propn If G is a Lie group and $u \in G$, then the centralizer $C_G(u)$ is a closed Lie subgroup, and its Lie algebra is $\{x \in \mathrm{Lie}(G) : \mathrm{Ad}(u)x = x\}$.

Pf To show $H = C_G(u)$ is a closed submfd of G, it suffices to show its intersection w/ a small nbhd of the identity is a closed sbmfd. In a nbhd N of the origin in $\mathrm{Lie}(G)$, the exponential map is a diffeomorphism onto $\exp(N)$, and we see that the preimage of $C_G(u)$ in N is a vector subspace by recalling that conjugation by u corresponds to the linear map $\mathrm{Ad}(u)$ of N. In particular $u \exp(tx) u^{-1} = \exp(t\,\mathrm{ad}(u)x)$, so $\exp(tx) \in C_G(u)$ for all $t \iff \mathrm{Ad}(u)x = x$.

Thm Let G be a ctd cpt Lie group and $S \subset G$ a torus. Then the centralizer $C_G(S)$ is a closed ctd Lie subgroup of G.

Pf Let $g \in C_G(S)$. By the previous propn, there exists an element $h \in C_G(S)$ which generates the closure H of the group generated by S and g. Let T be a maximal torus in G containing h. Then T centralizes S, so the closure of $T \cdot S$ is a ctd cpt abelian group, hence a torus, and by maximality of T we must have $S \subseteq T$. Now clearly $T \subseteq C_G(S)$, and since T is ctd, $T \subseteq C_G(S)^\circ$. Now $g \in H \subseteq T \subseteq C_G(S)^\circ$. Ergo $C_G(S)^\circ = C_G(S)$, i.e. $C_G(S)$ is connected. Now if u is a generator of $C_G(S)$, then $C_G(S) = C_G(u)$, so by the last thm $C_G(S)$ is a closed ctd Lie subgroup. $\square$