Recall let \( M \) be a Riemannian manifold of dimension \( n \); \( U \) a submanifold of dimension \( n-1 \) which is homeomorphic to a disk. We choose a unit normal in a fixed direction near \( U \). Geodesic coordinates near \( U \) are \((t, x_1, \ldots, x_n)\) such that \( t \mapsto (t, x_1, \ldots, x_n) \) is the geodesic through \( x \) in the direction of the unit normal.

Prop. In geodesic coordinates, \( g_{ii} = 0 \) for \( 2 \leq i \leq n \) and \( g_{11} = 1 \).

Proof. Recall that a geodesic should locally have
\[
\frac{d^2 x_k}{d t^2} = -\varepsilon_{ij} k^j \frac{d x_i}{d t} \frac{d x_j}{d t}.
\]
In our case \( \frac{d x_i}{d t} = 0 \) for \( i \neq 1 \), so \( \frac{d^2 x_k}{d t^2} = 0 \) for all \( k \). Ergo \( 0 = -\varepsilon_{1j} k^j \frac{d x_1}{d t} \frac{d x_j}{d t} = -\varepsilon_{11} k^1 \), so each \( -\varepsilon_{11} k^3 = 0 \). Given that \( (g_{ij}) \) is invertible, we see that \( [1, 1] = 0 \) as well. So \( \frac{1}{2} \left( \frac{\partial g_{11}}{\partial x_1} + \frac{\partial g_{11}}{\partial x_1} - \frac{\partial g_{11}}{\partial x_k} \right) = 0 \Rightarrow \frac{\partial g_{11}}{\partial x_1} = \frac{1}{2} \frac{\partial g_{11}}{\partial x_k} \) for all \( k \).

If we take \( k = 1 \), \( \frac{\partial g_{11}}{\partial x_1} = \frac{1}{2} \frac{\partial g_{11}}{\partial x_1} \Rightarrow \frac{\partial g_{11}}{\partial x_1} = 0 \), so for fixed \( x_2, \ldots, x_n \), \( g_{11} \) is a constant. When \( x_1 = 0 \), the initial condition of the geodesic through \((0, x_2, \ldots, x_n)\) is that it is tangent to the unit normal to the surface, so in particular its tangent vector \( \frac{\partial}{\partial x_1} \) at this point has length one. Ergo \( g_{11} = 1 \) when \( x_1 = 0 \) hence \( g_{11} = 1 \) throughout the geodesic coordinate neighborhood. Now consider \( 2 \leq k \leq n \) and \( \frac{\partial g_{11}}{\partial x_1} = \frac{1}{2} \frac{\partial g_{11}}{\partial x_k} = 0 \) since \( g_{11} \) is constant.

Now for \( x_1 = 0 \), by assumption \( \frac{\partial}{\partial x_1} \) and \( \frac{\partial}{\partial x_k} \) are orthogonal, so
9k vanishes when $x_i = 0$, so it vanishes for all $x_i \neq 0$.

We can now check that short geodesics are paths of shortest length:

**Prop. 1** Let $p : [0, 1] \to M$ be a geodesic. Then there exists an $\varepsilon > 0$ such that the restriction of $p$ to $[0, \varepsilon]$ is the unique path of shortest length from $p(0)$ to $p(\varepsilon)$.

**Prop. 2** Let $x \in M$. There exists a neighborhood $\mathcal{N}$ of $x$ such that for all $y \in \mathcal{N}$ there is a unique path of shortest distance from $x$ to $y$, which is a geodesic.

**Proof** We choose a hyperplane $V$ orthogonal to $p$ at $t = 0$ and construct geodesic coordinates as previously. Choose $\varepsilon$ and $\delta$ sufficiently small that the set $\mathcal{N}$ of points with coordinates 

$$\begin{align*}
&\delta \not\in \mathbb{R}^n \quad \text{for all } t \in [0, \varepsilon], \\
&0 < l_1 < \ldots < l_n < \delta
\end{align*}$$

is contained in the interior of the geodesic coordinate neighborhood. We assume that the coordinates of $p(t)$ are $(0, \ldots, 0)$, so that $p(0) = (0, \ldots, 0)$. Then $l_1 = \varepsilon$, where we let $l_1$ denote the length of the restriction of $p$ to $[0, \varepsilon]$.

Now we check that if $q : [0, \varepsilon] \to M$ is any admissible path with $q(0) = p(0)$ and $q(\varepsilon) = p(\varepsilon)$, then $l_q \geq l_1$. First, suppose that $q([0, \varepsilon])$ lies entirely within the neighborhood $\mathcal{N}$ and the $x_1$-coordinate of $q(t)$ is monotonically increasing. Reparametrizing $q$ we can arrange that $q(t)$ and $p(t)$ have the same $x_1$ coordinate, namely $t$. We then write $q(t) = (t, x_2(t), \ldots, x_n(t))$. Since $g_{1k} = g_{k1} = 0$

and $g_{11} = 1$, we see that
\[ l^2 = \int_0^\infty \sqrt{\sum_{i,j} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} \, dt \]

\[ = \int_0^\infty \sqrt{1 + \sum_{i,j} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} \, dt \]

Since \((g_{ij})\) is positive definite, its minor \((g_{ij})_{i,j \leq n}\) is also positive definite, so in particular

\[ \sum_{i,j} g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} > 0, \]

implying that

\[ l^2 \geq \int_0^\infty \sqrt{1} \, dt = \infty \Rightarrow l \geq 1. \]

**Exercise** Extend this argument to paths which have nonmonotonic first coordinate, or which are long enough to go outside of the neighborhood \(N\).

For part (ii), given a unit tangent vector \(x \in T_xM\), there is a unique geodesic \(p_x: [0, \varepsilon_x] \to M\) through \(x\) which is tangent to \(x\), if \(\varepsilon_x > 0\) chosen small enough that this is a path of shortest length.

Indeed, we can choose \(\varepsilon_x\) small enough that it also works for nearby unit tangent vectors \(y\). (To see this, pick a local diffeomorphism of \(M\) moving \(x\) to \(y\) regarded \(x\) as fixed while the metric varies; this produces small variations of the \(g_{ij}\), and the \(\varepsilon\) in part (i) can be used to work for sufficiently small variation of the \(g_{ij}\).

So for each unit vector \(x \in T_xM\) there is an \(\varepsilon_x > 0\) and a nbhd \(N_x\) of \(x\) in the unit ball of \(T_xM\) at \(p_x: [0, \varepsilon_x] \to M\) is a path of shortest length for all \(y \in N_x\). Since the unit ball in \(T_xM\) is compact, it can be covered by finitely many of the nbhds \(N_x\). Letting \(\varepsilon\) be the minimum of the corresponding \(\varepsilon_x\), we can take \(N\) to be the set of points connected to \(x\) by a geodesic of length \(\leq \varepsilon\).
If $M$ is a connected Riemannian manifold, we make $M$ into a metric space by defining $d(x,y)$ to be the infimum of $|p|$, where $p$ is a path from $x$ to $y$.

**Thm** Let $M$ be a compact connected Riemannian manifold, and let $x$ and $y$ be points of $M$. Then there is a geodesic $p: [0,1] \to M$ with $p(0) = x$ and $p(1) = y$.

**Remark** The condition in the theorem is stronger than it needs to be; this is true for manifolds that are geodesically complete, i.e., any well-paced geodesic can be extended to $(-\infty, \infty)$.

**PF** Let $\{p_i\}$ be a sequence of well-paced paths from $x$ to $y$ satisfying $d(p_i) \to d(x,y)$. Because they are well-paced, if $0 \leq a \leq b \leq 1$ we have that $d(p_i(a), p_i(b)) = (b-a) |p_i|_1$ and it follows that the $p_i$ are equicontinuous. Hence there is a subsequence that converges uniformly to a path $p$. This is clearly a geodesic (it's not immediately obvious that it's smooth). But on each sufficiently short interval $0 \leq a \leq b \leq 1$, $p(b)$ is near enough to $p(a)$ that the unique path of shortest length between them is a geodesic. It follows that $p$ itself is a geodesic.

Now we can consider applications to Lie groups.

**Thm** Let $G$ be a compact Lie group. There is a Riemannian metric on $G$ which is invariant under left and right translation. In this metric, any geodesic is a (right or left) translate of a path $t \mapsto \exp(tx)$ for some $x \in G$.
Since $G$ is a compact group acting by $Ad$ on the real vector space $g$, there is an $Ad(g)$-inner product on $g$. This gives us an inner product on $T_eG$, which we transfer to $T_gG$ by the left-transformation map. Right-translation in principle induces a different isomorphism $T_eG \rightarrow T_gG$, but these two maps differ by the derivative of conjugation, namely $Ad(g) : g \rightarrow g$, so since our inner product is invariant under $Ad(g)$, the invariant metric thusly defined is left under both left and right translation.

To show that any geodesic is a translate of the exponential map, it suffices to check that any short segment of a geodesic is of the form $t \rightarrow \exp(tx)$, but the metric is translation-invariant, so this just asks us to check that geodesics near the identity are of the form $t \rightarrow \exp(tx)$.

**Case I** $G \cong \mathbb{R}^n/\mathbb{Z}$ is a torus, and $T_eG \cong \mathbb{R}^n$ can be assumed to have the standard inner product. Then geodesics are just quotients of straight lines in $\mathbb{R}^n$ and therefore translates of the exponential map.

**Case II** For general $G$, let $x \in g$ and $Ex : (-\epsilon, \epsilon) \rightarrow G$ be the geodesic through the origin tangent to $x \in g$. For $t \in \mathbb{R}$, $t \mapsto Ex(tv)$ is the geodesic through the origin tangent to $DV$, so that $E(xv) : E_{tv}(x)$. So there is a small $U$ of the origin in $g$ and a map $E : U \rightarrow G$ $st. Ex(t) = E(tx)$ for $x, tv \in U$. Wts this is the same as the exponential map.
Given \( g \in G \), then \( \text{trans} \) \( (E(tg)) \) on the left by \( g \) and the
right by \( g^{-1} \) gives a one-parameter, one-point homogeneous to \( \text{Ad}(g)X \).
Thus if \( (x, t) \in \mathcal{V} \), \( \mathcal{G}E(tx)g^{-1} = E(t\text{Ad}(g)x) \). For \( \text{Exp} \) \( (x) \), let \( T \) be a
maximal torus containing the one-parameter subgroup \( \text{exp}(tx) \), \( t \in \mathbb{R}^n \).
Then \( E(tx) \) commutes with \( g \) when \( t \) \( \in \mathcal{U} \). Since the path \( t \mapsto E(tx) \)
runs through the centralizer \( \mathcal{C}(T) \) and the centralizes through \( \mathbb{N}(T) \).
Recalling that the closed component of \( \mathbb{N}(T) \) containing the identity
is 1, we see that \( E(tx) \) lies in \( T \). But the geodesics in \( T \) are
exactly \( \exp (tx) \). \( \square \)

**Thm:** Let \( G \) be a Lie group, and \( g \) its Lie algebra.
Then the exponential map \( \exp : g \to G \) is surjective.

**PF:** Put a translation-invariant Riemannian structure on \( G \). Then given
\( g \in G \), there is a geodesic path from the identity to \( g \). But this
paths of the form \( t \mapsto \exp(tx) \).

**Thm:** Let \( G \) be a Lie group and \( T \) a maximal torus.
There exists \( k \in G \), \( \forall \) \( g \in T \).

**PF:** Let \( g \) and \( t \) be the Lie algebras of \( G \) and \( T \), then \( t \)
be a generator of \( T \). Let \( x \in g \) and \( t \) be such that \( \text{exp}(t) \)
and \( e^t \) coincide, which is possible since \( e^t \) is unique since \( g \) is compact.
Since \( G \) is a compact group acting by \( \text{Ad} \) on \( G \), there is
inner product \( \langle \cdot, \cdot \rangle \) on \( g \). Choose \( x \in G \) so that
the real value \( \langle x, \text{Ad}(t)x \rangle \) is maximal. (This maximum is
achieved, by compactness of \( G \).) Let \( t = \text{Ad}(k)x \). Then
$\exp(H) = e^H$ is a generator for $eTK^{-1}$.

If $Y$ is arbitrary, then $\langle x, \text{Ad}(e^{tY})H \rangle_{e^{\mathfrak{h}}} = \langle x, \text{Ad}(Y)H \rangle = \langle x, [H, Y] \rangle = -\langle x, [H, Y] \rangle$.

Since the bilinear form is invariant under $\text{Ad}$, it is in fact under $e^t$, meaning that $0 = \langle x, [H, Y] \rangle + \langle [H, X], Y \rangle$. So $\langle [H, X], Y \rangle = 0$ for all $Y$. Therefore $\exp(H)$ commutes with $\exp(tX)$ for all $t \in \mathbb{R}$.

Since $\exp(H)$ generates $eTK^{-1}$ the one-parameter subgroup $\exp(tX)$ is contained in the centralizer of $eTK^{-1}$. But $eTK^{-1}$ is in fact a maximal torus, so it follows that $\exp(tX) \subseteq eTK^{-1}$.

In particular, $g \exp(tX) \subseteq eTK^{-1}$.

Thus (Cartan) let $G$ be a compact connected Lie group and $T$ be a maximal torus. Then every maximal torus is conjugate to $T$, and every element of $G$ is contained in a conjugate of $T$.

PF The second statement is implied by the first one. To show the first statement, let $T'$ be another maximal torus and $g$ a generator. Then $eTK^{-1}$ is contained in $eT'K^{-1}$ for some $K$. As both are maximal tori, they are equal.

Proof Let $G$ be a connected compact Lie group, $S \subseteq G$ a torus (not necessarily maximal), and $g \in G(s)$ an element of $S$.

By $S$ and $g$, then $H$ has a topological generator that is in $S$.
Since $H$ is closed and abelian, its connected component $H^0$ containing the identity is a torus. Let $h_0$ be a generator of the torus. Now $H/H^0$ is compact and discrete, hence finite. Since $SS H^0$ and $S$ and $g$ together generate a dense subgroup of $H$, the finite group $H/H^0$ is cyclic and generated by $gH^0$. Let $r$ be the order of this group, so $g^r \in H^0$. Since the $r$th power map $H^0 \to H^0$ is surjective, if $u \in H^0$ then $(gu)^r = h_0$. Then the group generated by $h_0$ contains $h_0$ and a generator $gH^0 = (gu)H^0$ of $H/H^0$. So $h$ is a topological generator of $H$. 

**Prop.** If $G$ is a Lie group and $u \in G$, then the centralizer $C_G(u)$ is a closed Lie subgroup, and its Lie algebra is $\mathfrak{g} \times \mathfrak{e}(u)$.

**Pf.** To show $H = C_G(u)$ is a closed submanifold of $G$, it suffices to show its intersection with a small nbhd of the identity is a closed submanifold. In a nbhd $N$ of the origin in $\mathfrak{g}$, the exponential map is a diffeomorphism onto $\exp(N)$, and we see that the preimage of $C_G(u)$ in $N$ is a vector subspace by recalling that conjugation by $u$ corresponds to the linear map $\text{Ad}(u)$ of $N$. In particular, $\exp(tx)u^{-1} = \exp(t \text{Ad}(u)x)$, so $\exp(tx) \in C_G(u)$ for all $t \in \mathbb{R}$.
Thm Let $G$ be a compact Lie group and $S \subset G$ a torus. Then the centralizer $C^c_G(S)$ is a closed compact Lie subgroup of $G$.

PF Let $g \in C^c_G(S)$. By the previous propn, there exists an element $h \in C^c_G(S)$ which generates the closure $H$ of the group generated by $S$ and $g$. Let $T$ be a maximal torus in $G$ containing $h$. Then $T$ centralizes $S$, so the closure of $T \cdot S$ is a compact abelian group, hence a torus, and by maximality of $T$ we must have $S \leq T$. Now clearly $T \leq C^c_G(S)$, and since $T$ is compact, $T \leq C^c_G(S)^o$. Now $g \in H \leq T \leq C^c_G(S)^o$. E.g. $C^c_G(S)^o = C^c_G(S)$, i.e. $C^c_G(S)$ is connected. Now if $u$ is a generator of $C^c_G(S)$, then $C^c_G(S) = C^c_G(u)$, so by the last thm $C^c_G(S)$ is a closed compact Lie subgroup.