Lecture 22

A brief aside on equivariant cohomology followed by work on conjugacy of maximal tori.

Suppose we have a cts action of a group G on a space $X \circlearrowleft G$.
Let's say X has the homotopy type of a CW cpx.
How do we incorporate into some kind of algebraic invt?

Idea 1 Take a quotient

- Probably destroys a lot of structure (eg quotient of a mfd might not be a mfd).
- Not functorial: A G-equivariant homotopy equivalence $X \to Y$ might not induce isomorphism on $H^*(X/G) \longrightarrow H^*(Y/G)$.

  Consider $\underset{\mathbb{Z}/2\mathbb{Z}}{\overset{S^\infty}{\circlearrowleft}} \longrightarrow \underset{\mathbb{Z}/2\mathbb{Z}}{\overset{*}{\circlearrowleft}}$; $H^*(\mathbb{R}P^\infty) \not\cong H^*(pt)$.

Idea 2 Make the action free, then take a quotient.

- EG weakly contractible space w/ a free G-action
- $X \times EG$ has free diagonal action
- The Borel construction is the balanced product $X \times_G EG$
- Equivariant cohomology is $H^*(X \times_G EG; R) = H^*_G(X; R)$

Examples

(1) $X \circlearrowleft G$ free $\Rightarrow H^*_G(X; R) = H^*(X/G; R)$

(2) $X = *$ w/ trivial action $\Rightarrow H^*_G(X; R) = H^*(BG; R)$

Properties

1. There is a fiber bundle $X \hookrightarrow X \times_G EG \to BG$, so $H^*(BG;R)$ acts on $H^*_G(X;R)$ via pullback and taking cup products. So $H^*_G(X;R)$ is a module over $H^*(BG;R)$. This extra structure is usually most interesting.

   eg $H^*_{\mathbb{Z}/2\mathbb{Z}}(X;\mathbb{F}_2)$ is always a module over $H^*(\mathbb{R}P^\infty;\mathbb{F}_2) = \mathbb{F}_2[\sigma]$

2. $X \longrightarrow Y$ G-equivariant map induces a map $H^*_G(X) \longrightarrow H^*_G(Y)$; an equivariant homotopy equivalence induces an isomorphism. (exercise)

3. There is a relationship w/ the fixed set X^{Fix}: If X is a finite-dimensional CW cpx w/ an action of $\mathbb{Z}/p\mathbb{Z}$, then $\dim(H^*(X;\mathbb{Z}/p\mathbb{Z})) \geq \dim(H^*(X^{Fix};\mathbb{Z}/p\mathbb{Z}))$.

   - This was proved by P. Smith in 1938 using long exact sequences. But the modern proof, essentially introduced by Borel, uses spectral sequences and an isomorphism between (localized) equivariant cohomology & the cohomology of the fixed set.

4. Remark: For G discrete, there is an induced action $G \curvearrowright C_*(X)$, and $H^*_G(X;\mathbb{F}) = \mathrm{Ext}_{\mathbb{F}[G]}(C_*,\mathbb{F})$. This gives a sense of how to generalize.

Conjugacy of maximal tori via geodesics (Ref: Bump Section 16)

Idea Deduce that any two maximal tori in a compact Lie group are conjugate via the surjectivity of the exponential map.

Remark If you prefer a more topological proof, poke me for a reference.

Recall

Defn A Riemannian manifold is a smooth mfd M w/ a smoothly-varying inner product g_x on each fiber of the tangent space T_x.

Smoothly varying = If $x_1, \dots, x_n$ are coordinates on $U \subseteq M$ w/ $U \simeq \mathbb{R}^n$, then we consider the basis for each $T_x M$ consisting of $\{\frac{\partial}{\partial x_i}\}$. Then the functions $g_{ij}(x) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$ are smooth on U.

Let (g^{ij}) be the inverse to the matrix (g_{ij}), so that $\sum_j g_{ij} g^{jk} = \delta_{ik}$. (And of course both matrices are symmetric.)

Defn A smooth path $p : [0,1] \to M$ is admissible if $p_*(\frac{d}{dt}) \neq 0$ at any point on $[0,1]$. The length or arclength of p is

$$|p| = \int_0^1 \left| p_*\left(\frac{d}{dt}\right) \right| dt$$

Which in local coordinates becomes

$$\left| p_*\left(\frac{d}{dt}\right) \right| = \sqrt{\sum_{i,j} g_{ij} \frac{\partial x_i}{\partial t} \frac{\partial x_j}{\partial t}}$$

A path is well-paced or parametrized by arclength if $\int_0^a \left| p_*\left(\frac{d}{dt}\right) \right| dt = |p| \cdot a$.

Quick to check via chain rule that reparametrization preserves arclength & every path has a unique well-paced reparametrization.

<u>Note</u> It's useful to use Einstein's summation notation, in which repeated indices indicate sums.

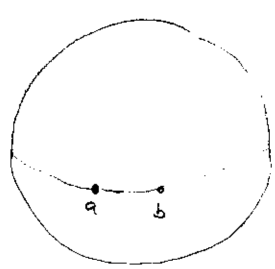
i.e. let
$$p : [0,1] \longrightarrow U \subseteq M$$
$$t \longmapsto (x_1(t), \dots, x_n(t))$$

So that
$$\frac{dF}{dt}(x_1(t), \dots, x_n(t)) = \sum_{i=1}^{n} \frac{dx_i}{dt} \frac{\partial F}{\partial x_i}(x_1(t), \dots, x_n(t))$$

We can rewrite this as $\frac{dF}{dt} = \frac{dx_i}{dt} \cdot \frac{\partial F}{\partial x_i}$.

<u>Defn</u> If for each smooth curve $q : [0,1] \longrightarrow M$ w/ the same endpoints as p we have $|p| \leq |q|$, then p is a <u>path of shortest length</u>.

Roughly speaking, a geodesic is a well paced path which is a path of shortest length on short intervals.



eg great circles on the sphere

Consider a smooth family of paths $[0,1]\times(-\varepsilon,\varepsilon) \to M$, $(t,u) \longmapsto p_u(t)$, where $p_u(0) = p_0(0)$, $p_u(1) = p_0(1)$.

Let $f(u) = |p_u|$. We say that p_0 is of <u>stationary length</u> if $f'(0) = 0$ for each such deformation. For example, if $p_0 = p$ is a path of shortest length, then $f(0)$ is a minimum so $f'(0) = 0$.

Given $x_1, \dots, x_n$ coordinate functions on $U \subseteq M$, we let the <u>Christoffel symbols</u> be $\Gamma_{kij} = [ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right)$. <u>of the first kind</u>

$$\Gamma^k_{ij} = g^{k\ell}\Gamma_{\ell ij} = g^{k\ell}[ij,\ell]$$

<u>of the second kind</u>

<u>Propn</u> Suppose that $p:[0,1] \to M$ is a well-paced admissible path. If $p(t)$ lies in U w/ coordinates $x_1, \dots, x_n$, then if $x_i(t) = x_i(p(t))$, p is of stationary length if and only if it satisfies

$$\frac{\partial^2 x_k}{\partial t^2} = -\Gamma^k_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt}$$

[Coordinate free description: $\nabla_{\dot\gamma}\dot\gamma = 0$, i.e. parallel transport along the curve preserves the tangent vector to the curve.]

<u>Pf</u> We consider the effects of deforming the path. Let $p_u(t)$ be a smooth deformation of p. Let $x_i(p(u,t))$ be the coordinate functions. Since $p_u(0) = p(0)$ and $p_u(1) = p(1)$, we have that $\frac{\partial x_i}{\partial u} = 0$ where $t=0$ or $t=1$. In local coordinates the arclength formula becomes

$$|p_u| = \int_0^1 \sqrt{g_{ij}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t}}\;dt$$

We assume $p(t) = p_0(t)$ is well-paced, so that the integrand is constant (that is, independent of t) when $u=0$. Let $F(u) = |p_u|$. We have

$$F'(t) = \frac{\partial}{\partial u}\int_0^1 \sqrt{g_{ij}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t}}\;dt$$

$$= \int_0^1 \left(g_{ij}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t}\right)^{-\frac{1}{2}}\left[\frac{1}{2}\,\frac{\partial g_{ij}}{\partial u}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t} + \frac{1}{2}\,g_{ij}\,\frac{\partial^2 x_i}{\partial u\,\partial t}\,\frac{\partial x_j}{\partial t} + \frac{1}{2}\,g_{ij}\,\frac{\partial x_i}{\partial t}\,\frac{\partial^2 x_j}{\partial u\,\partial t}\right]dt$$

chain rule.

equal, remember $g_{ij} = g_{ji}$

$$= \int_0^1 \left(g_{ij}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t}\right)^{-\frac{1}{2}}\left[\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x_\ell}\,\frac{\partial x_\ell}{\partial u}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t} + g_{ij}\,\frac{\partial^2 x_i}{\partial u\,\partial t}\,\frac{\partial x_j}{\partial t}\right]dt$$

We integrate the second term by parts to get

$$\begin{array}{c|c} g_{ij}\,\frac{\partial x_j}{\partial t} & \frac{\partial^2 x_i}{\partial u\,\partial t} \\ \hline \frac{\partial}{\partial t}\left(g_{ij}\,\frac{\partial x_j}{\partial t}\right) & \frac{\partial x_i}{\partial u} \end{array}$$

$=0$ when $u=0$

$$F'(0) = \int_0^1 \left(g_{ij}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t}\right)^{-\frac{1}{2}}\left[\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x_\ell}\,\frac{\partial x_\ell}{\partial u}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t} - \frac{\partial x_i}{\partial u}\,\frac{\partial}{\partial t}\left(g_{ij}\,\frac{\partial x_j}{\partial t}\right)\right]dt$$

$$= \int \left(g_{ij}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t}\right)^{-\frac{1}{2}}\left[\frac{1}{2}\,\frac{\partial g_{ij}}{\partial x_\ell}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t} - \frac{\partial}{\partial t}\left(g_{\ell j}\,\frac{\partial x_j}{\partial t}\right)\right]\frac{\partial x_\ell}{\partial u}\,dt\,.$$

Note that the partials $\frac{\partial x_\ell}{\partial u}$ are arbitrary except that they vanish at $t=0$ and $t=1$. So p is of stationary length $\Leftrightarrow$ $F'(0) = 0$ $\Leftrightarrow$

$$0 = \frac{1}{2}\,\frac{\partial g_{ij}}{\partial x_\ell}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t} - \frac{\partial}{\partial t}\left(g_{\ell j}\,\frac{\partial x_j}{\partial t}\right)$$

that is, if

$$g_{\ell j}\,\frac{\partial^2 x_j}{\partial t^2} = \frac{1}{2}\,\frac{\partial g_{ij}}{\partial x_\ell}\,\frac{\partial x_i}{\partial t}\,\frac{\partial x_j}{\partial t} - \frac{\partial g_{\ell j}}{\partial t}\,\frac{\partial x_i}{\partial t}\,.$$

But the last term expands as

$$\frac{\partial g_{ej}}{\partial t}\frac{\partial x_j}{\partial t} = \frac{\partial g_{ej}}{\partial x_i}\frac{\partial x_i}{dt}\frac{\partial x_j}{\partial t}$$

$$= \frac{1}{2}\left[\frac{\partial g_{ej}}{\partial x_i} + \frac{\partial g_{ei}}{\partial x_j}\right]\frac{\partial x_i}{dt}\frac{\partial x_j}{dt}$$

So the condition at the bottom of the preceding page becomes

$$g_{ej}\frac{\partial^2 x_j}{\partial t^2} = \frac{1}{2}\left[\frac{\partial g_{ij}}{\partial x_e}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial t}\right] - \frac{1}{2}\left[\frac{\partial g_{ej}}{\partial x_i} + \frac{\partial g_{ei}}{\partial x_j}\right]\frac{\partial x_i}{dt}\frac{\partial x_j}{dt}$$

$$= -\frac{1}{2}\left[\frac{\partial g_{ej}}{\partial x_i} + \frac{\partial g_{ei}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_e}\right]\frac{\partial x_i}{dt}\frac{\partial x_j}{dt}$$

$$= -[ij,e]\,\frac{\partial x_i}{\partial t}\cdot\frac{\partial x_j}{\partial t}$$

$$\text{So } \frac{\partial^2 x_k}{\partial t^2} = -g^{ke}[ij;e]\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial t} = -\{ij,k\}\frac{\partial x_i}{\partial t}\frac{\partial x_j}{\partial t}. \quad \square$$

A <u>geodesic</u> is a solution to the differential equations $\frac{d^2 x_k}{dt^2} = -\{ij,k\}\frac{dx_i}{dt}\frac{dx_j}{dt}$.

<u>Propn</u> Let x be a point on a Riemannian manifold M, and $X \in T_x M$. Then for sufficiently small ε, there is a unique geodesic $p: (-\varepsilon,\varepsilon) \longrightarrow M$ such that $p(0) = x$ and $p_*\left(\frac{d}{dt}\right) = X$.

<u>Pf</u> Let $x_1, \dots, x_n$ be the coordinate functions, and $y_1, \dots, y_n$ be a set of new variables, and rewrite the desired conditions as

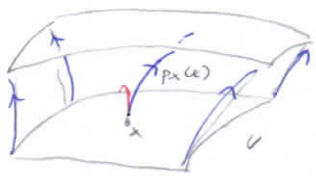
$$\frac{dx_i}{dt} = y_i$$

$$\frac{dy_k}{dt} = -\{ij,k\}\,y_i y_j$$

Then $p(0) = x$ and $p_*\left(\frac{d}{dt}\right) = X$ give initial conditions for the system, and existence and uniqueness follow from general theory of first-order diff eq.

Geodesic Coordinates

Say we have a hypersurface (smooth submfd of codimn 1). Then



we can look at geodesics at $x \in U$ w/ tangent vector a unit normal vector. Indeed, we can translate U along each of these geodesics – that is, each $x \mapsto p_x(1)$ to get a hypersurface U'. The interesting thing is that this is <u>also</u> normal to the geodesics.

To prove this (and subsequently prove that sufficienty small geodesics are shortest paths) we want <u>geodesic coordinates</u>.

Let $x_2, \dots, x_n$ be local coordinates on U. We as that the path $t \mapsto (t, x_2, \dots, x_n)$ agree w/ the geodesic $p_x(t)$.

<u>Propn</u> In geodesic coordinates, $g_{1i} = 0$ for $2 \leq i \leq n$, and $g_{11} = 1$.

This amounts to saying that geodesic curves w/ tangent vector $\frac{\partial}{\partial x_1}$ are orthogonal to the level hypersurfaces $x_1 = 0$ w/ tangent spaces spanned by $\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}$

<u>Next time</u> Proof, application to lengths of geodesics, & consequences for Lie groups.