Lecture 21

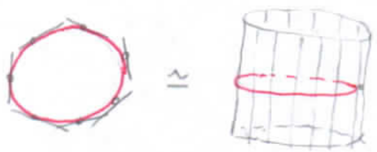
Principal Bundles Ctd: Relationship to vector bundles & classification

Recall A vector bundle $p: E \longrightarrow B$ is a fiber bundle w/ fiber some vector space V and structure group $GL(V)$. (I.e., every point $x \in B$ has a neighborhood U st $p^{-1}(U) \simeq U \times V$ and the transition maps are linear on each fiber.)

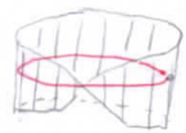
Examples

denotes the trivial bundle

$TS^1 \simeq \varepsilon$



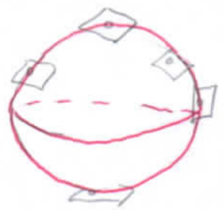
Mobius band



Note There is a natural notion of the sum of bundles $E \to B$, $E' \to B$ via pullbacks

$$\begin{tikzcd} E \oplus E' \arrow[r] \arrow[d] & E \times E' \arrow[d] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd}$$

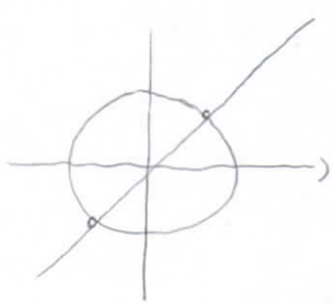
TS^2



Not isomorphic to $S^2 \times \mathbb{R}^2$

$NS^2 \subseteq T\mathbb{R}^3$

The canonical line bundle $\gamma = \gamma^1_{n,\mathbb{R}}$ over $\mathbb{RP}^n$ is the bundle (L, v) where L is a line in $\mathbb{R}^{n+1}$ and v is a vector on the line.



Similarly for $\gamma^1_{n,\mathbb{C}}$ and $\mathbb{CP}^n$.

These are the canonical line bundles.

Exercise $\gamma^1_{1,\mathbb{R}}$ is the Mobius band

To a vector bundle $\xi = (E, p, B)$ we associate a principal bundle as follows: let $P = P_\xi = V_n \xi$ where $V_n \xi$ is the n-frame bundle; that is, the set of pairs $(x, \underline{v})$ with $x \in B$ and $\underline{v} = (v_1, \ldots, v_n)$ is an n-frame in $p^{-1}(x) = E_x \simeq V$. (This is topologized as a subset of $nE = E \oplus \cdots \oplus E$.) The group $GL_n(\mathbb{R})$ acts on the right via $(v_1, \ldots, v_n)A = (v_1 A, \ldots, v_n A)$. This makes P into a principal $GL_n(\mathbb{R})$-bundle over B.

<u>Alternately</u> Let $\underline{\mathrm{Hom}}(\varepsilon^n, E)$ be the bundle consisting in each fiber of linear maps $\mathbb{R}^n \longrightarrow E_x$ (note that this is different from the space of vector bundle morphisms.) This can be identified w/ $E^{\oplus n}$. $GL_n(\mathbb{R})$ acts on the left of ε^n by $g \cdot (x, v) = (x, g \cdot v)$, which induces a <u>right</u> action on $\underline{\mathrm{Hom}}(\varepsilon^n, E)$. We can set $P_\xi = \underline{\mathrm{Iso}}(\varepsilon^n, E) \subseteq \underline{\mathrm{Hom}}(\varepsilon^n, E)$. Easy to see this is the same bundle.

<u>Propn</u> For any base B, there is a natural bijection

$$\phi : P_{GL_n(\mathbb{R})} B \xrightarrow{\sim} \mathrm{Vect}_n^{\mathbb{R}} B$$

$\sim$ Isomorphism classes of vector bundles

given by $P \longmapsto P \times_{GL_n \mathbb{R}} \mathbb{R}^n$. The inverse is $\xi \longmapsto P_\xi$. Likewise for $\mathbb{C}$.

Defn A Euclidean metric on a real vector bundle $E \to B$ is a map $q: E \to \mathbb{R}$ st the restriction to each fiber is positive definite & quadratic (and therefore induces an inner product in the usual way). Likewise for a Hermitian metric on a complex vector bundle.

If $\xi = (E, p, B)$ has a Euclidean metric, we may define an associated principal $O(n)$-bundle using $V_n^o \xi$ the space of orthonormal frames. Likewise a complex bundle w/ a Hermitian metric has an associated principal $U(n)$-bundle.

Lemma Let ξ be a real vector bundle w/ two Euclidean metrics q_0, q_1. Then the associated principle $O(n)$-bundles are isomorphic. Similarly complex, Hermitian, $U(n)$.

Ergo

every open cover has a locally finite refinement

Bundles over a paracompact Hausdorff space always admit a Euclidean/Hermitian metric

Propn For any paracompact Hausdorff space B, there are natural bijections

$$\mathcal{P}_{O(n)} B \xrightarrow{\sim} \mathrm{Vect}_n^{\mathbb{R}}(B)$$

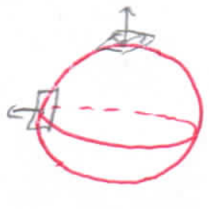
$$\mathcal{P}_{U(n)} B \xrightarrow{\sim} \mathrm{Vect}_n^{\mathbb{C}}(B)$$

Examples

① $\gamma = \gamma^1_{n,\mathbb{R}}$ canonical line bundle over $\mathbb{R}P^n$.
The associated principal $O(1) = \mathbb{Z}/2\mathbb{Z}$-bundle P_γ is the 2:1 covering map $S^n \to \mathbb{R}P^n$, and $S^n \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{R} \cong E$

② Likewise if $\gamma = \gamma^1_{n,\mathbb{C}}$, the associated principal $U(1)$-bundle is the quotient map $S^{2n+1} \to \mathbb{CP}^n$, w/ $U(1) \simeq S^1$ acting by complex multiplication. So $S^{2n+1} \times_{S^1} \mathbb{C} \simeq E$.

③ The orthonormal frame bundle of S^n is exactly $V_{n+1}, \mathbb{R}^{n+1} = O(n+1)$



So the associated principal bundle is the quotient map $O(n+1)/O(n)$

④ A k-dim'l distribution (as defined when we talked about the local Frobenius thm) is a section of the bundle $G_k TM$ where fiber at $x \in M$ is the Grassmanian of k-planes in T_xM. This is expressible as $G_k TM = P_{TM} \times_{GL_n(\mathbb{R})} G_k \mathbb{R}^n$.

⇝ Following this example one gets a functorial construction of all sorts of fiber bundles associated to a vector bundle (projective bundles, sphere bundles, disc bundles)

<u>Equivariant maps & sections</u>

<u>Example</u> How would we construct a section of $\gamma^1_{n,\mathbb{R}}$?

$$\begin{array}{c}\gamma^1_{n,\mathbb{R}} \\ \downarrow \\ \mathbb{RP}^n\end{array}$$

One way is to pick a function $F: S^n \to \mathbb{R}^n$ which is $\mathbb{Z}/2\mathbb{Z}$ equivariant, i.e. $F(-x) = -F(x)$. Then recalling that $E = \{(L, v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : v \in L\}$, we let $g : S^n \longrightarrow E$, $x \longmapsto ([x], F(x)\cdot x)$, which factors through $S^n \to \mathbb{RP}^n$.

So we get a section
$$\begin{tikzcd} E \arrow[d, "\pi"'] \\ \mathbb{RP}^n \arrow[u, bend left, "s"] \end{tikzcd}$$
. Indeed, any section can be obtained this way, since if we have a section s we can let $g = s\pi$, and then $g(x) = ([x], F(x)x)$ for some function F w/ $F(-x)\cdot -x = F(x)\cdot x$.

<u>Rephrase</u> There is a bijective correspondence between the set of $\mathbb{Z}/2\mathbb{Z}$-equivariant maps $S^n \longrightarrow \mathbb{R}$ and sections of $S^n \times_{\mathbb{Z}/2\mathbb{Z}} \mathbb{R} \longrightarrow S^n/(\mathbb{Z}/2\mathbb{Z})$. Written this way we have a generalization:

<u>Propn</u> Let P be a principal G-bundle over B and X a right G-space. There is a natural bijection between the set of equivariant maps $\mathrm{Hom}_G(P, X)$ and the set of sections $\Gamma(P \times_G X \longrightarrow B)$, where given $F \in \mathrm{Hom}_G(P, X)$ we construct a section s_F via considering the equivariant map
$$P \longrightarrow P \times X, \quad p \longmapsto (p, F(p))$$
and passing to G-orbits.

<u>PF</u> For $P \simeq B \times G$ trivial, $\mathrm{Hom}_G(B\times G, X) = \mathrm{Hom}(B, X)$ the set of cts maps, and $\Gamma(B \times G \times_G X) \longrightarrow B = \Gamma(B \times X \to B) = \mathrm{Hom}(B, X)$. For the general case, if $\{U_i\}$ is a cover of B w/ $P_i = P|_{U_i}$ trivial, one can do a diagram chase w/

$$\begin{tikzcd}
\mathrm{Hom}_G(P,X) \arrow[r] \arrow[d, "\phi"] & \prod \mathrm{Hom}_G(P_i,X) \arrow[r] \arrow[d, "\phi'"] & \prod \mathrm{Hom}_G(P_i \cap P_j, X) \arrow[d, "\phi''"] \\
\Gamma(P \times_G X \to B) \arrow[r] & \prod \Gamma(P_i \times_G X, U_i) \arrow[r] & \prod \Gamma((P_i \cap P_j) \times_B X \to U_i \cap U_j)
\end{tikzcd}$$

which is exact in the sense that an element $\{f_i\} \in \prod \mathrm{Hom}_G(P_i, X)$ with the property that

$$\begin{tikzcd}
\mathrm{Hom}_G(P_i,X) \ni f_i \arrow[dr, mapsto] \arrow[r] & \\
 & g_{ij} \in \mathrm{Hom}(P_i \cap P_j, X) \\
\mathrm{Hom}_G(P_j,X) \ni f_j \arrow[ur, mapsto] \arrow[r] &
\end{tikzcd}$$

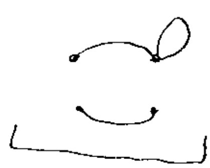
For each pair (i,j)

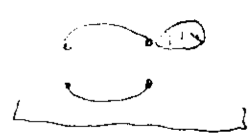
is the image of a map in $\mathrm{Hom}_G(P,X)$, and likewise for the bottom row. Then if ϕ' bijective and ϕ'' injective, ϕ must be bijective.

Classifying Spaces

Recall A CW complex X is a space built out of closed disks by specifying attaching maps along their boundaries.


0-skeleton X^0 is copies of D^0

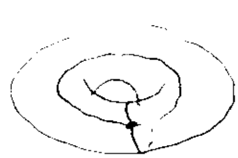

1-skeleton X^1 attaches copies of D^1


2-skeleton X^2 attaches copies of D^2

$$X^n = X^{n-1} \cup_{e^n_\alpha} (\cup D^n_\alpha)$$

where $e_n^\alpha : \partial D^n_\alpha \longrightarrow X^{n-1}$

eg CW-decomposition of the torus



Defn A Serre Fibration is a cts map $\pi : E \to B$ which has the homotopy lifting property for CW cpxes X, i.e.

$$\begin{tikzcd} X \arrow[r, "\hat{f}_0"] \arrow[d, "X\times\{0\}"'] & E \arrow[d, "\pi"] \\ X \times I \arrow[r, "F"'] \arrow[ur, dashed] & B \end{tikzcd}$$

Fiber bundles are examples of Serre fibrations (or more to the point, the concept of fibrations generalizes fiber bundles).

Propn Let $P \to X$ be a principal G-bundle and let $f, g : B \to X$ be homotopic maps. Then the pullbacks f^*P and g^*P are isomorphic as principal G-bundles over B.

Indeed if we take the homotopy $F : B \times I \to X$ and consider F^*P, all we need is

Lemma Let $Q \to B \times I$ be a principal G-bundle and Q_0 its restriction to $B \times \{0\}$. Then Q is isomorphic to $Q_0 \times I$. In particular $Q_0 \simeq Q_1$.

Pf Suffices to construct a morphism $Q \to Q_0 \times I$, or equivalently to constructing a section of $Q \times_G (Q_0 \times I) \to B \times I$. Certainly we have a section over $B \times \{0\}$. One can check using the homotopy lifting property that this section extends over $B \times I$. $\square$

So we have $B \mapsto \mathcal{P}_G(B)$ is a homotopy functor from CW-cpxes to sets; homotopy equivalences induce bijections.

Corollary If B is contractible every principle G-bundle over B is trivial.

Note that both these things are clearly also true of vector bundles.

Recall A space is weakly contractible if $X \to *$ is a weak equivalence (i.e., X has trivial homotopy groups).

Thm Let $P \to B$ be a principal G-bundle w/ P weakly contractible. Then for all CW cpxes X, the map $\phi : [X, B] \to \mathcal{P}_G(X)$ given by $f \mapsto f^*P$ is bijective.

In this situation we say B is a classifying space and P is a universal G-bundle.

Pf Suppose that P is weakly contractible. Then we first show that ϕ is onto. Say that $Q \to B$ is a principal G-bundle. Then $Q \times_G P \to B$ is a Serre fibration w/ weakly contractible fiber. One can show (exercise) that this implies that it admits a section, and therefore that there is a G-equivariant map $\tilde{f} : Q \to P$. Let $f : B \to X$ be the induced map on orbit spaces. Then $Q \cong f^*P$ as requested.

Now suppose we have maps $f_0, f_1 : B \to X$ and an isomorphism $\psi : f_0^*P \to f_1^*P$. Let Q be the principal G-bundle $f_0^*P \times I$ over $B \times I$, and consider the product $p : Q \times_G P \to B \times I$. There is a section of p over $(B \times \{0\}) \cup (B \times \{1\})$; since the fiber is weakly contractible, this section extends over all of $B \times I$, so it determines a G-equivariant map $Q \to P$. Passing to orbit spaces gives a homotopy $B \times I \to X$ from f_0 to f_1. So ϕ is injective. $\square$

Propn Suppose a universal G-bundle $P \longrightarrow B$ exists. Then

ⓐ B can be taken to be a CW-cpx

ⓑ B is unique up to (canonical) homotopy equivalence

ⓒ P is unique up to G-homotopy equivalence.

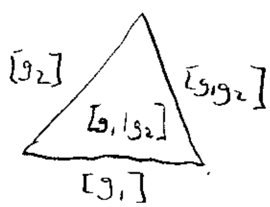
Pf Part ⓐ is because any space B has a weak equivalence $f: B' \longrightarrow B$ from a CW-cpx B' and we may pull back along f.
Parts ⓑ and ⓒ are fiddling w/ category theory.

Classifying spaces exist

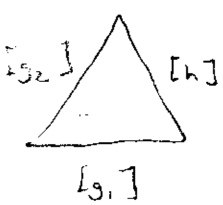
For G a discrete group, let EG be the simplicial space w/ n-simplices G^{n+1}, i.e. an n-simplex is $[g_0, \dots, g_n]$ w/ boundaries $\partial_i[g_0, \dots, g_n] = [g_0, \dots, \hat{g_i}, \dots, g_n]$. G acts freely by $g \cdot [g_0, \dots, g_n] = [gg_0, \dots, gg_n]$. $BG = EG/G$ ($EG \xrightarrow{p} BG$) has n-simplices

$[g_1 | g_2 | \dots | g_n] = p([1, g_1, g_1 g_2, \dots, g_1 \cdots g_n])$ w/

$$\partial_0 [g_1 | \dots | g_n] = [g_2 | \dots | g_n]$$
$$\partial_n [g_1 | \dots | g_n] = [g_1 | \dots | g_{n-1}]$$
$$\partial_i [g_1 | \dots | g_n] = [g_1 | \dots | g_i g_{i+1} | \dots | g_n]$$



More generally if G is not discrete a two simplex is



and a continuous path from $g_1 g_2$ to h in G.

<u>Example</u> $G = \mathbb{Z}/2\mathbb{Z}$

$$\begin{tikzcd} EG = S^\infty \arrow[d] \\ BG = \mathbb{RP}^\infty \end{tikzcd}$$

<u>Example</u> If G is discrete, $K(G,1)$ is BG.

<u>Example</u> If $G = GL_n(\mathbb{R})$, then

$$\begin{tikzcd} V_n\mathbb{R}^\infty \arrow[d] \\ G_n\mathbb{R}^\infty \end{tikzcd}$$

is a universal $GL(n,\mathbb{R})$ bundle.

Likewise for

$$\begin{tikzcd} V_n^o\mathbb{R}^\infty \arrow[d] \\ G_n\mathbb{R}^\infty \end{tikzcd}$$

, so that $BO(n) \simeq G_n\mathbb{R}^\infty$. Likewise $BU(n) \simeq G_n\mathbb{C}^\infty$.

This gives a nice (and very useful) classification of vector bundles.

<u>Example</u> Complex line bundles are classified by maps $X \longrightarrow \mathbb{CP}^\infty$. (As it happens these are exactly equivalent to $H^2(X)$.)