Principal Bundles 21: Relationship to vector bundles & classification

Recall A vector bundle $p: E \rightarrow \mathcal{B}$ is a fiber bundle with fiber some vector space $V$ and structure group $GL(V)$. (i.e., every point $x \in \mathcal{B}$ has a neighborhood $U$ so $p^{-1}(U) \cong U \times V$ and the transition maps are linear on each fiber.)

Examples

$T^2 \times \mathbb{R}$

Notes: There is a natural notion of the sum of bundles $E \rightarrow \mathcal{B}, E' \rightarrow \mathcal{B}$ via pullbacks:

$E \oplus E' \rightarrow E \times E'$

$\text{Not isomorphic to } \mathbb{R}^2 \times \mathbb{R}^2$

The canonical line bundle $\gamma = \gamma_{\mathbb{R}}$ over $\mathbb{R}^{1}$ is the bundle $(L, V)$ where $L$ is a line in $\mathbb{R}^{n+1}$ and $V$ is a vector on the line.

Similarly for $\gamma^1$ and $\mathbb{CP}^{n}$.

These are the canonical line bundles.

Exercise $\gamma^1_{\mathbb{R}}$ is the Mobius band.
To a vector bundle $E = (E, p, \mathcal{B})$ we associate a principal bundle as
follows: let $P = \mathcal{P} \equiv V^\otimes E$ where $V^\otimes E$ is the $n$-frame bundle, that
is, the set of pairs $(x, y)$ with $x \in \mathcal{B}$ and $y = (v_1, \ldots, v_n)$ is an
$n$-frame in $p^{-1}(x) = E_x \cong \mathbb{R}^n$. (This is topologized as a subset of
$n\mathcal{B} = E \otimes \cdots \otimes E$.) The group $GL_n(\mathbb{R})$ acts on the right via
$(v_1, \ldots, v_n)A = (Av_1, \ldots, Av_n)$. This makes $P$ into a principal $GL_n(\mathbb{R})$-bundle
over $\mathcal{B}$.

Alternatively let $\text{Hom}( Em, E)$ be the bundle consisting in each
fiber of linear maps $\mathbb{R}^m \to E_x$ (note that this is different from
the space of vector bundle morphisms.) This can be identified with $E \otimes \mathbb{R}^m$.

$GL_n(\mathbb{R})$ acts on the left of $\mathbb{R}^m$ by $g(x, y) = (x, gy)$, which induces
a right action on $\text{Hom}( Em, E)$. We can set $P = \text{iso} (Em, E) \times \text{Hom}( Em, E)$.

Easy to see this is the same bundle.

Proof. For any base $e$, there is a natural bijection

$$
\phi: \text{iso} (Em, e) \times \text{Hom}( Em, e) \to V_e^\otimes E
$$

a isomorphism classes of

vee e bundles

Given by $p \mapsto px \in \mathbb{R}^n$. The inverse is $E \mapsto P$. Likewise for $\mathcal{B}$. 
A Euclidean metric on a real vector bundle $E \to B$ is a map $q : E \to \mathbb{R}$ at the restriction to each fiber is positive definite, quadratic (and therefore induces an inner product in the usual way). Likewise for a Hermitian metric on a complex vector bundle.

If $E = (E, p, \theta)$ has a Euclidean metric, we may define an associated principal $O(n)$-bundle using $V^n$ the space of orthonormal frames. Likewise a complex bundle with a Hermitian metric has an associated principal $U(n)$-bundle.

Lemma: Let $E$ be a real vector bundle with two Euclidean metrics $g, g'$. Then the associated principal $O(n)$-bundles are isomorphic. Similarly complex, Hermitian, $U(n)$.

Ergo: every open cover has a locally finite refinement

Prop: For any paracompact Hausdorff space $B$, there are natural bijections

$$\text{P} \quad \theta \longrightarrow \text{Vect}_n^\mathbb{R}(\theta)$$

$$\text{P} \quad \theta \longrightarrow \text{Vect}_n^\mathbb{C}(\theta)$$

Examples:

1. $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ canonical line bundle over $\mathbb{R}P^n$.

The associated principal $O(1)$-bundle $P_\theta$ is the 2:1 covering map $S^n \longrightarrow \mathbb{R}P^n$, and $S^n \times \mathbb{R} \subset E$. 

\[ \text{def} \]
2. Likewise if $\gamma = \gamma^n$, the associated principal $U(1)$-bundle is the quotient map $S^{2n+1} \to \mathbb{C}P^n$, with $U(1) = S^1$ acting by complex multiplication. So $S^{2n+1} / S^1 \cong E$.

3. The orthonormal frame bundle of $S^n$ is exactly $V_{n-1}, \mathbb{R}^{n-1} = O(n-1)$, so the associated principal bundle is the quotient map $O(n+1) / O(n)$.

4. A $k$-dim'l distribution (as defined when we talked about the local Frobenius thm) is a section of the bundle $G_kTM$ whose fiber at $x \in M$ is the Grassmanian of $k$-planes in $T_x M$. This is expressible as $G_kTM = \bigtimes_{x \in M} G_k \mathbb{R}^n$.

Following this example one gets a functorial construction of all sorts of fiber bundles associated to a vector bundle (projective bundles, sphere bundles, disc bundles).

Equivariant maps & sections

Example How would we construct a section of $\gamma^n / \mathbb{R}$?

One way is to pick a function $F : S^n \to \mathbb{R}^n$ which is $\mathbb{Z}/2\mathbb{Z}$ equivariant, i.e., $F(-x) = -F(x)$. Then recalling that $E = \mathbb{R}(L, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1}$, so $L$.
we let \( g: S^n \to E \), which factors through \( S^n \to \mathbb{R}P^n \) via \( x \mapsto ([x], f(x) \cdot x) \).

So we get a section \( E \). Indeed, any section can be obtained this way, since if we have a section \( s \) we can let \( g = s \pi_1 \), and then \( g(x) = (\pi_1, f(x) \cdot x) \) for some function \( f \) with \( f(-x) \cdot x = f(x) \cdot x \).

**Rephrase** There is a bijective correspondence between the set of \( \mathbb{Z}/2 \mathbb{Z} \)-equivariant maps \( S^n \to \mathbb{R} \) and sections of \( S^n \to \mathbb{R}P^n \).

Written this way we have a generalization:

**Prop** Let \( P \) be a principal \( G \)-bundle over \( \Theta \) and \( X \) a right \( G \)-space. There is a natural bijection between the set of equivariant maps \( \text{Hom}_G(P, X) \) and the set of sections \( \Gamma(P \times_X \Theta \to \Theta) \), where given \( f \in \text{Hom}_G(P, X) \) we construct a section \( s \) via considering the equivariant map \( P \to P \times_X \Theta \) and passing to \( G \)-orbits, \( s(p) = (p, f(p)) \).

**Proof** For \( P = \Theta \times_G \Theta \) trivial, \( \text{Hom}_G(\Theta \times_G \Theta, X) = \text{Hom}(\Theta, X) \) the set of cts maps, and \( \Gamma(\Theta \times_G \Theta \to \Theta) = \Gamma(\Theta \times_X \Theta \to \Theta) = \text{Hom}(\Theta, X) \). For the general case, if \( \Sigma U \Theta \) is a cover of \( \Theta \) w/ \( P_i \to P_i \) trivial, one can do a diagram chase w/
\[ \text{Hom}_G(P; x) \rightarrow \prod \text{Hom}_G(P_i; x) \rightarrow \prod \text{Hom}_G(P_i \cap P_j; x) \]

which is exact in the sense that an element \( \delta f_i \in \text{Hom}_G(P_i; x) \) with the property that \( \text{Hom}_G(P_i; x) \) for each pair \( (i, j) \)

is the image of a map in \( \text{Hom}_G(P; x) \), and likewise for the bottom row. Then \( \phi' \) is bijective and \( \phi'' \) is injective, \( \phi \) must be bijective.

**Classifying Spaces**

Recall a CW complex \( X \) is a space built out of closed disks by specifying attaching maps along their boundaries.

- 0-skeleton \( X^0 \) is copies of \( P^0 \)
- 1-skeleton \( X^1 \) attaches copies of \( P^1 \)
- 2-skeleton \( X^2 \) attaches copies of \( P^2 \)

\[ X^n = X^{n-1} \cup (U \times D^n) \]

where \( e_n : D^n \rightarrow X^{n-1} \)

\[ e_n \circ \partial \]

\( e_n \circ \partial \) is the boundary of the torus.
**Defn** A **Serre Fibration** is a c.s.s. map \( \pi : E \to B \) which has the homotopy lifting property for CW complexes \( X \), i.e., \[
\begin{array}{ccc}
\pi & \longrightarrow & E \\
\uparrow \quad & & \uparrow \ \
\downarrow \ \\
X \times S^0 & \longrightarrow & \ast \\
\downarrow \ \\
X \times I & \longrightarrow & B
\end{array}
\]

Fiber bundles are examples of Serre fibrations (or more to the point, the concept of fibrations generalizes fiber bundles).

**Propn** Let \( P \to X \) be a principal \( G \)-bundle and let \( f, g : B \to X \) be homotopic maps. Then the pullbacks \( f^*P \) and \( g^*P \) are isomorphic as principal \( G \)-bundles over \( B \).

Indeed if we take the homotopy \( F : \pi \times I \to X \) and consider \( F^*P \), all we need is

**Lemma** Let \( Q \to B \times I \) be a principal \( G \)-bundle and \( Q_0 \) its restriction to \( B \times \{0\} \), then \( Q \) is isomorphic to \( Q_0 \times I \). In particular \( Q_0 \cong Q_1 \).

**PF** Suffices to construct a morphism \( Q \to Q_0 \times I \), or equivalently to constructing a section of \( Q \tilde{\times} (Q_0 \times I) \to B \times I \). Certainly we have a section over \( B \times \{0\} \). One can check using the homotopy lifting property that this section extends over \( B \times I \). \(\square\)

So we have \( B \to P_\pi (B) \) is a homotopy functor from CW-complexes to sets; homotopy equivalences induce bijections.

**Corollary** If \( B \) is contractible every principle \( G \)-bundle over \( B \) is trivial.

Note that both these things are clearly also true of vector bundles.
Recall a space is weakly contractible if $X \to *$ is a weak equivalence (i.e., $X$ has trivial homotopy groups).

Let $P \to \mathcal{G}$ be a principal $G$-bundle w/ $P$ weakly contractible. Then for all CW complexes $X$, the map $\mathcal{G} \times [X, \mathcal{G}] \to P \times (x)$ given by $P \to f^*P$ is bijective.

In this situation we say $\mathcal{G}$ is a classifying space and $P$ is a universal $G$-bundle.

Proof: Suppose that $P$ is weakly contractible. Then we first show that $\mathcal{G}$ is onto. Say that $Q \to \mathcal{G}$ is a principal $G$-bundle. Then $Q \times_P P \to \mathcal{G}$ is a Serre fibration w/ weakly contractible fibers. One can show (exercise) that this implies that it admits a section, and therefore that there is a $G$-equivariant map $f : Q \to P$. Let $f : \mathcal{G} \to X$ be the induced map on orbit spaces. Then $Q \times f^*P$ as requested.

Now suppose we have maps $f_0, f : \mathcal{G} \to X$ and an isomorphism $\psi : f_0^*P \to f^*P$. Let $\alpha$ be the principal $G$-bundle $f_0^*P \times I$ over $\mathcal{G} \times I$, and consider the product $p : \mathcal{G} \times P \to \mathcal{G} \times I$. There is a section of $P$ over $(\mathcal{G} \times \{0\}) \cup (\mathcal{G} \times \{1\})$; since the fiber is weakly contractible, this section extends over all of $\mathcal{G} \times I$, so it determines a $G$-equivariant map $Q \to P$. Passing to orbit spaces gives a homotopy $\mathcal{G} \times I \to X$ from $f_0$ to $f$. So $\alpha$ is injective. $\square$
Suppose a universal $G$-bundle $P \to \Theta$ exists. Then

1. $\Theta$ can be taken to be a CW-complex (canonical)
2. $\Theta$ is unique up to homotopy equivalence
3. $P$ is unique up to $G$-homotopy equivalence.

**Proof**

Part 1 is because any space $\Theta$ has a weak equivalence $P \to \Theta$ and we may pull back along $P$.

Parts 2 and 3 are tiding with category theory.

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**Classifying spaces exist**

For $G$ a discrete group, let $EG$ be the simplicial space w/ n-simplices $G^n$, i.e. an n-simplex is $[g_0, \ldots, g_n]$ w/ boundaries $\partial^n [g_0, \ldots, g_n] = [g_0, \ldots, ^g g_i, \ldots, g_n]$. $G$ acts freely by

$G \cdot [g_0, \ldots, g_n] = [g g_0, \ldots, g g_n]$. $BG = EG/G$ has n-simplices

$[g_0, \ldots, g_n] = p([g_0, g_1, g_2, \ldots, g_n])$ w/ $\partial^n [g_0, \ldots, g_n] = [g_0, \ldots, ^g g_i, \ldots, g_n]$,

$\partial^n [g_0, g_1, \ldots, g_{n-1}] = [g_1, \ldots, g_n]$, $\partial^n [g_0, g_1, \ldots, g_{n-1}] = [g_1, \ldots, g_n]$.

More generally if $G$ is not discrete a two simplex is

and a continuous path from $g_0$ to $g_2$ in $G$. $[h] \to [g_0, g_2]$.
Example \[ G = \mathbb{Z}/2\mathbb{Z} \] \[ \text{EG} = S^\infty \]
\[ \text{BG} = \mathbb{RP}^\infty \]

Example: If \( G \) is discrete, \( K(G,1) \) is \( BG \).

Example: If \( G = \text{GL}_n(\mathbb{R}) \), then \( V_n \mathbb{R}^\infty \) is a universal \( \text{GL}_n(\mathbb{R}) \) bundle.
\[ \downarrow \]
\[ G_n \mathbb{R}^\infty \]

Likewise for \( V_n \mathbb{IR}^\infty \), so that \( G\mathbb{O}(n) \approx G_n \mathbb{IR}^\infty \). Likewise
\[ G\mathbb{U}(n) \approx G_n \mathbb{C}^\infty. \]

This gives a nice (and very useful) classification of vector bundles.

Example: Complex line bundles are classified by maps \( X \longrightarrow G\mathbb{P}^\infty \).
(As it happens, these are exactly equivalent to \( H^2(X) \).)