Intro to Principal Bundles

Recall $G 	o M$ smoothly, freely, and properly, then $M/G$ is a manifold and $\pi ^{-1}(U) \cong U \times G$

Locally we have $\pi ^{-1}(U) \cong U \times G$

This is an example of a principal bundle:

**Defn:** A fiber bundle w/ fiber $F$ is a map $p: E \to B$ w/ the property that any point $x \in B$ has a neighborhood $U \subset B$ for which there exists a homeomorphism $\varphi _x : F \times U \to p^{-1}(U)$ for which $p \circ \varphi _x = \pi _x$ the projection map onto $U$. (last condition is "over $U_x$")

**Remark:** This generalizes the notion of a product $F \times B \to B$.

**Nontrivial example**

1. The Mobius band over the circle

2. Any fiber bundle whose fiber $F$ is a discrete space is a covering map

Say $p: E \to B$ is a fiber bundle w/ fiber $F$. Then we can pick an open cover $U = \{U_i\}$ on which we have a $\varphi _i : E_{U_i} : F \times U \to p^{-1}(U_i)$ over $U_i$ on overlaps $U_{ij} = U_i \cap U_j$ we have maps $F \times U_{ij} \xrightarrow{\varphi _i \times 1} p^{-1}(U_{ij})$. 
Let the maps $E_{ij}$ satisfy a cocycle condition: 
\[
\begin{align*}
E_{ij} E_{jk} &= E_{ik} \\
E_{ii} &= \text{id}
\end{align*}
\]

For fixed $x \in U_{ij}$, the map $E_{ij}(\cdot, x)$ is a homeomorphism $F \cong F$, and we can alternately write $E_{ij} : U_{ij} \rightarrow \text{Homeo}(F)$. Assuming $F$ is locally compact, $E_{ij}$ or $E_{ij}$ is.

**Special cases** $E_{ij}$ land in a specific subgroup of $\text{Homeo}(F)$.

**Defn** Let $G \leq \text{Homeo}(F)$. (That is, $G$ is a group acting faithfully on $F$.) A fiber bundle $p : E \rightarrow B$ w/ fiber $F$ and structure group $G$ is a fiber bundle for which there exists a trivialization $\mathcal{U}$ for which the corresponding cocycle $E_{ij} : U_{ij} \rightarrow \text{Homeo}(F)$ factors as $U_{ij} \xrightarrow{E_{ij}} \text{Homeo}(F)$.

These $E_{ij}$ satisfy the cocycle condition.

**Example** Let $V$ be a real or complex vector space. Then a vector bundle over a (locally compact) space $B$ is a fiber bundle w/ fiber $V$ and structure group $GL(V)$.

(Err, the transition maps $E_{ij}$ are linear on the fibers.)

**Example** The tangent bundle $TM$ of a real $n$-dimensional manifold $M$ is an $n$-dimensional vector bundle over $M$. (The transition maps are the derivatives of the coordinate change maps.)
Let $G$ be a topological group. A principal $G$-bundle is a fiber bundle $p: P \to B$ with fiber $G$ and structure group $G$, where $G$ acts on itself by left-translation. Hence a principal $G$-bundle has trivializing maps

$$
\phi_i : G \times U_i \to p^{-1}(U_i)
$$

at the cocycle $\phi_{ij} : G \times U_{ij} \to G \times U_{ij}$ is of the form

$$
\phi_{ij}(g, x) = (\psi_{ij}(x) \cdot g, x)
$$

for some map $\psi_{ij} : U_{ij} \to G$.

**Exercise** A quotient map $M$ of a free proper action is a principal $G$-bundle.

**Alternate characterizations**

A principal $G$-bundle is a locally trivial free $G$-space $P$ with orbit space $B$.

**Proof** A morphism of principal $G$-bundles $P \to B$ and $Q \to B$ is a map $P \to Q$ so $p \circ \sigma \to q$ and $\sigma$ commutes with the action of $G$ on either side.
Naturally we say that a morphism is an \textit{isomorphism} if it admits an inverse in the same category.

**Prop.** Every morphism of principal $G$-bundles is in fact an isomorphism.

**Remark.** This illustrates how different this notion is from a fiber bundle.

Let $P = G = B \times G$ be actual products. Then $\sigma(x, g) = (x, f(x)g)$ for some $f \in F_B$. For $F : G \to G$, $\sigma$ is an isomorphism with inverse given by $\sigma^{-1}(x, g) = (x, F(x)^{-1}g)$. Since every principal bundle is locally trivial, this in fact proves the proposition in general.

**Prop.** A principal $G$ bundle is trivial if and only if it admits a section.

**PF.** If $P \to B$ is trivial, it admits a section by $s : B \times G \to P$. Conversely, let $s : B \to P$ be a section. Then $\phi : B \times G \to P$

$$\quad (x, g) \mapsto s(x) \cdot g$$

is a morphism of principal bundles, hence an isomorphism. So $P$ is trivial.

**Remark.** This is incredibly untrue of vector bundles, which always possess a section via inclusion $X \times \mathbb{R}^n$. (Whether a vector bundle has an everywhere nonzero section is more interesting but still doesn't imply triviality.)
Notation Let $P_B$ be the set of isomorphism classes of principal $G$-bundles over $B$.

More examples

1. We already have $G \times M$, $M$ is principal $G$-bundle. Notice that this includes $H \leq G$ a closed subgroup, so that $G \to G/H$ is a (smooth) principal $H$-bundle. So, e.g., the bundles we considered in February are all examples: $O(n) \to O(n)/O(n_1)$ is a principal $\tilde{O}(n)$ bundle over $O(n)/O(n_1) \cong S^n$.

2. $G$ discrete $\implies$ a principal $G$-bundle is exactly a regular covering map with $G$ as the group of deck transformations.

3. Frame bundles of vector-bundles are see next section.
Pullbacks and (balanced) products

Given a principal $G$-bundle $P \rightarrow \mathcal{B}$ and a map $f : \mathcal{B}' \rightarrow \mathcal{B}$, the pullback in the space $P' = f^*P = \mathcal{B}' \times \mathcal{B} = \mathcal{B}' \times \pi(P) ; f(x') = \pi(P)$

$$f^*P \rightarrow P$$
$$\downarrow \quad \downarrow$$
$$\mathcal{B}' \rightarrow \mathcal{B}$$

This inherits a natural structure of a principal $G$-bundle over $\mathcal{B}'$.

Remarks

1. Sections of the pullback bundle $f^*P$ are lifts of $\{ \text{lifts of } P \}$

\[ \begin{array}{c}
\mathcal{B}' \quad \mathcal{B} \\
\downarrow \quad \downarrow \\
P \quad P
\end{array} \]

2. If $P$ is a principal $G$-bundle over $\mathcal{B}'$, bundle maps $\mathcal{B}' \rightarrow f^*P$ are in one-to-one correspondence with commutative squares

\[ \begin{array}{c}
\mathcal{B}' \quad \mathcal{B} \\
\downarrow \quad \downarrow \\
\mathcal{B}' \quad \mathcal{B}
\end{array} \]

which the top arrow is $G$-equivariant.

Remark: If we pull back $P$ over itself we get a bundle

\[ \begin{array}{c}
P \times P \rightarrow P \\
\downarrow \quad \downarrow \\
P \quad \mathcal{B}
\end{array} \]

where $\pi'$ is projection on the lefthand factor and the action of $G$ on the bundle is on the righthand factor. Then the diagonal map is a section, so this bundle is always trivial.

(This fact is of some importance in algebraic geometry.)
Recall A left $G$-action can always be converted to a right $G$-action by setting $xg = g^{-1}x$, and vice versa.

**Definition** If $W$ is a right $G$-space and $X$ is a left $G$-space, the balanced product $W 	imes_G X$ is the quotient $W 	imes X /\sim$, where $\sim$ is the relation $((w,x) \sim (w',x'))$ if there exists $g \in G$ such that $(w,x)g = (w',x')$. The quotient by the diagonal action $(w,x)g = (w,g^{-1}x)$.

**Special cases**

1. $X = *$ is a point $\Rightarrow W \times * \cong W / G$.
2. $X = G$ with left translation $\Rightarrow W \times G \cong W$ homeomorphism.

If $G$ and $H$ are topological groups, a $(G,H)$-space $Y$ is a space with a left $G$-action and right $H$-action such that the actions commute; that is, $(gy)h = g(wh)$. If $Y$ is a $(G,H)$-space and $X$ is a right $G$-space, then $X \times_Y Y$ is a right $H$-space via $(x,yh) = (x,yh)$; similarly if $\tilde{z}$ is a right $H$-space for $Y \times Z$.

**Exercise** Let $X$ be a right $G$-space, $Y$ a $(G,H)$-space, and $Z$ a left $H$-space. There is a natural homeomorphism $(x \times_Y Y) \times Z \cong X \times (Y \times Z)$.

**Remark** The tricky part is continuity of the obvious map.

So we can omit parentheses to write $y \times x \times' \times z \cong x \times' y \times z$.

**Corollary 1** Suppose $X$ is a right $G$-space and $Y$ is a left $H$-space. For $H$ a subgroup of $G$. Then $X \times_H G \times Y \cong X \times Y$.

**Corollary 2** Let $X$ be a right $G$-space and $H$ be a subgroup of $G$. Then $X \times (G/H) \cong X / H$. 


Q If \( X \to X/G \) is a principal \( G \)-bundle, is \( X \to X/H \) necessarily a principal \( H \)-bundle?

A No, let \( G = (\mathbb{R}, \times) \) acting on itself, and \( H = \mathbb{R}^+ \). \( \mathbb{R} \to \mathbb{R}/\mathbb{R}^+ \) is not locally trivial.

Para A subgroup \( H \) of \( G \) is admissible if \( G \to G/H \) is a principal \( H \)-bundle.

Para Let \( P \to B \) be a principal \( G \)-bundle, and \( H \) an admissible subgroup of \( G \). Then the quotient map \( P \to P/H \) is a principal \( H \)-bundle.

PF For any subgroup \( H \), we have \( P/H = P \times_B (G/H) \). Then \( P \to P/H \) is the bundle \( P \times_B (G/H) \). Since \( G \to G/H \) is principal we are done.

If we fix a left \( G \)-space \( W \), the map \( X \to W \times X \) is functorial.

Let \( \pi: P \to B \) be a principal \( G \)-bundle and \( F \) a left \( G \)-space. The constant map \( F \to * \) is \( G \)-equivariant, hence induces \( P \times F \to P \times * = \emptyset \).
This is a fiber bundle \( \emptyset \times F \) over \( B \) and structure group \( G \).

Exercise Any fiber bundle \( F \to B \) with structure group \( G \) can be reconstructed from a principal bundle using this operation.

Exercise This operation commutes w/ the pullback: that is, if \( F: B \to \mathcal{B} \) then there is a natural homeomorphism \( F^*(P \times F) \cong (E \times F) \times F \).

Next Time Relationship between vector bundles & principle bundles, classifying spaces.