Lecture 20

Intro to Principal Bundles

Recall $G \curvearrowright M$ smoothly, freely, & properly, then M/G is a manifold and

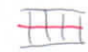
$$\begin{tikzcd} M \arrow[d, "\pi"] \\ M/G \end{tikzcd}$$ π is smooth. Locally we have $$\begin{tikzcd} \pi^{-1}(U) \simeq U \times G \arrow[d] \\ U \end{tikzcd}$$

This is an example of a principal bundle.

Also both spelled "fibre"

<u>Defn</u> A <u>Fiber bundle</u> w/ Fiber F is a map $p: E \to B$ w/ the property that any point $x \in B$ has a nbhd $U \subseteq B$ for which there exists a homeomorphism $\varphi_U : F \times U \xrightarrow{\sim} p^{-1}(U)$ for which $p \circ \varphi_U = \pi_2$ the projection map onto U. (last condition = "over U_i")

<u>Remark</u> This generalizes the notion of a product $F \times B \to B$.

<u>Nontrivial example</u> ① The Möbius band over the circle  $U \times I$ 

② Any fiber bundle whose fiber F is a discrete space is a covering map

Say $p: E \to B$ is a fiber bundle w/ fiber F. Then we can pick an open cover $\mathcal{U} = \{U_i\}$ on which we have a $\varphi_i = \varphi_{U_i} : F \times U_i \to p^{-1}(U_i)$ over U_i. On overlaps $U_{ij} = U_i \cap U_j$ we have maps

$$\begin{tikzcd} F \times U_{ij} \arrow[r, "\varphi_j", "\sim"'] \arrow[dr, dashed] & p^{-1}(U_{ij}) \\ & F \times U_{ij} \arrow[u, "\varphi_i"', "\wr"] \end{tikzcd}$$

st the maps φ_{ij} satisfy a "cocycle condition" $\begin{cases} \varphi_{ij}\varphi_{jk} = \varphi_{ik} \\ \varphi_{ii} = id \end{cases}$

For fixed $x \in U_{ij}$, the map $\varphi_{ij}(-,x)$ is a homeomorphism $F \xrightarrow{\sim} F$, and we can alternately write $\overline{\varphi_{ij}} : U_{ij} \to \mathrm{Homeo}(F)$. Assuming F is locally compact, φ_{ij} cts $\Leftrightarrow$ $\overline{\varphi_{ij}}$ is.

Special cases $\overline{\varphi_{ij}}$ land in a specific subgroup of Homeo(F).

Defn Let $G \leq \mathrm{Homeo}(F)$. (That is, G is a group acting faithfully on F.) A fiber bundle $p: E \to B$ w/ fiber F and structure group G is a fiber bundle for which there exists a trivialization $\mathcal{U}$ for which the corresponding cocycle $\overline{\varphi_{ij}} : U_{ij} \to \mathrm{Homeo}(F)$ factors as

$$\begin{tikzcd} U_{ij} \arrow[r, "\overline{\varphi_{ij}}"] \arrow[dr, "\psi_{ij}"'] & \mathrm{Homeo}(F) \\ & G \arrow[u, hook] \end{tikzcd}$$

These ψ_{ij} satisfy the cocycle condition.

Example Let V be a real or complex vector space. Then a vector bundle over a (locally compact) space B is a fiber bundle w/ fiber V and structure group $GL(V)$.

(Ie, the transition maps φ_{ij} are linear on the fibers.)

Example The tangent bundle TM of a real n-dim'l mfd M is an n-dim'l vector bundle over M. (The transition maps are the derivatives of the coordinate change maps.)

Defn Let G be a topological group. A principal G-bundle is a fiber bundle $p: P \to B$ w/ fiber G and structure group G, where G acts on itself by left-translation.. Hence a principal G-bundle has trivializing maps

$$\varphi_i : G \times U_i \longrightarrow p^{-1}(U_i)$$

st the cocycle $\varphi_{ij} : G \times U_{ij} \longrightarrow G \times U_{ij}$ is of the form

$$\varphi_{ij}(g, x) = (\psi_{ij}(x) \cdot g, x)$$

for some map $\psi_{ij} : U_{ij} \longrightarrow G$.

Exercise A quotient map
$$\begin{tikzcd} M \arrow[d] \\ M/G \end{tikzcd}$$
of a free proper action is a principal G-bundle.

Alternate characterizations

A principal G-bundle is a locally trivial free $\underbrace{G\text{-space}}_{\substack{\text{topological space}\\ \text{w/ an action of}\\ G \text{ (on the right, in}\\ \text{this case.)}}}$ P w/ orbit space B.

Defn A morphism of principal G-bundles $P \to B$ and $Q \to B$ is a map $P \xrightarrow{\sigma} Q$ st
$$\begin{tikzcd} P \arrow[rr, "\sigma"] \arrow[dr] & \arrow[d, phantom, "G"] & Q \arrow[dl] \\ & B & \end{tikzcd}$$
and σ commutes w/ the action of G on either side.

Naturally we say that a morphism is an <u>isomorphism</u> if it admits an inverse in the same category.

<u>Propn</u> Every morphism of principal G-bundles is in fact an isomorphism.

<u>Remark</u> This illustrates how different this notion is from a fiber bundle.

<u>Pf</u> Suppose first that $P = Q = B \times G$ are actual products. Then $\sigma(x, g) = (x, F(x) \cdot g)$ for some cts fcn $F: B \longrightarrow G$. Ergo σ is an isomorphism w/ inverse given by $\sigma^{-1}(x, g) = (x, F(x)^{-1} g)$. Since every principal bundle is locally trivial, this in fact proves the proposition in general.

<u>Propn</u> A principal G bundle is trivial if and only if it admits a section.

<u>Pf</u> If $P \longrightarrow B$ is trivial, it admits a section by $\times \{e\} \hookrightarrow B \times G = P$. Conversely let $s: B \longrightarrow P$ be a section. Then

$$\begin{aligned} \phi: B \times G &\longrightarrow P \\ (x, g) &\longmapsto s(x) \cdot g \end{aligned}$$

is a morphism of principal bundles, hence an isomorphism. So P is trivial.

<u>Remark</u> This is incredibly untrue of vector bundles, which always possess a section via inclusion $X \times \{0\}$. (Whether a vector bundle has an everywhere nonzero section is more interesting but still doesn't imply triviality.)

<u>Notation</u> Let $\mathcal{P}_G B$ be the set of isomorphism classes of principal G-bundles over B.

<u>More examples</u>

① We already have $G \curvearrowright M$, $\begin{array}{c} M \\ \downarrow \\ M/G \end{array}$ is principal G-bundle. Notice

that this includes $H \leq G$ a closed subgroup, so that $G \longrightarrow G/H$ is a (smooth) principal H-bundle. So, eg, the bundles we considered in February are all examples: $O(n) \hookrightarrow \begin{array}{c} O(n+1) \\ \downarrow \\ O(n+1)/O(n) \simeq S^n \end{array}$ is a principal

$O(n)$ bundle over

② G discrete $\Rightarrow$ a principal G-bundle is exactly a regular covering map w/ G as the group of deck transformations.

$\mathbb{Z}/3\mathbb{Z}$-principal bundle

③ Frame bundles of vector-bundles $\rightsquigarrow$ see next section.

Pullbacks and (balanced) products

Given a principal G-bundle $P \xrightarrow{\pi} B$ and a c/s map $f: B' \longrightarrow B$, the pullback is the space $P' = f^*P = B' \times_{\times} P = \{(x', p) : f(x') = \pi(p)\}$

$$\begin{tikzcd} f^*P \arrow[r] \arrow[d] \arrow[dr, phantom, "G"] & P \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$$

This inherits a natural structure of a principal G-bundle over X'.

Remarks ① {Sections of the pullback bundle} $\longleftrightarrow$ $\left\{ \text{Lifts of } \begin{tikzcd} & P \arrow[d] \\ B' \arrow[r, "f"'] \arrow[ur, dashed] & B \end{tikzcd} \right\}$

② If Q is a principal G bundle over X', bundle maps $Q \longrightarrow f^*P$ are in one-to-one correspondence w/ commutative squares $\begin{tikzcd} Q \arrow[r] \arrow[d] & P \arrow[d] \\ B' \arrow[r] & B \end{tikzcd}$ in which the top arrow is G-equivariant.

Remark If we pullback P over itself we get a bundle

$$\begin{tikzcd} P \times_{\times} P \arrow[r] \arrow[d, "\pi'"'] & P \arrow[d, "\pi"] \\ P \arrow[r, "\pi"'] \arrow[u, dashed, bend right, "\Delta"'] & B \end{tikzcd}$$

where π' is projection on the lefthand factor and the action of G on the bundle is on the righthand factor. Then the diagonal map is a section, so this bundle is always trivial.

(This fact is of some import in algebraic geometry.)

Products

Recall A left G-action can always be converted to a right G-action by setting $xg = g^{-1}x$, and vice versa.

Defn If W is a right G-space and X is a left G-space, the balanced product $W \times_G X$ is the quotient $W \times X / (wg, x) \sim (w, gx)$, or equivalently the quotient by the diagonal action $(w,x)g = (wg, g^{-1}x)$.

Special cases ① $X = *$ is a point $\Rightarrow W \times_G * = W/G$.

② $X = G$ w/ left translation $\Rightarrow W \times_G G \xrightarrow{\sim} W$ homeomorphism.

If G and H are topological groups, a (G,H)-space Y is a space w/ a left G-action and right H-action such that the actions commute: that is, $(gy)h = g(yh)$. If Y is a (G,H)-space and X is a right G-space, then $X \times_G Y$ is a right H-space via $(x,y)h = (x, yh)$; similarly if Z is a right H-space for $Y \times_H Z$.

Exercise Let X be a right G-space, Y a (G,H)-space, and Z a left H-space. There is a natural homeomorphism $(X \times_G Y) \times_H Z \simeq X \times_G (Y \times_H Z)$.

Remark The tricky part is continuity of the obvious map.

So we can omit parentheses to write eg $X \times_G Y \times_H Z \simeq X \times_G Y \times_H Z$.

Corollary 1 Suppose X is a right G-space and Y is a left H-space for H a subgroup of G. Then $X \times_G G \times_H Y \simeq X \times_H Y$.

Corollary 2 Let X be a right G-space and H be a subgroup of G. Then

$$X \times_G (G/H) \simeq X/H.$$

Q If $X \to X/G$ is a principal G bundle, is $X \to X/H$ necessarily a principal H-bundle?

A No, let $G = (\mathbb{R}, +)$ acting on itself, and $H = \mathbb{Q}$. $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$ is not locally trivial.

Defn A subgroup H of G is <u>admissible</u> if $G \to G/H$ is a principal H-bundle.

Propn Let $P \to B$ be a principal G-bundle, and H an admissible subgroup of G. Then the quotient map $P \to P/H$ is a principal H-bundle.

Pf For any subgroup H, we have $P/H = P \times_G (G/H)$. Then $P \to P/H$ is the bundle $P \times_G G \to P \times_G (G/H)$. Since $G \to G/H$ is principal we are done.

If we fix a left G-space W, the map $X \mapsto W \times_G X$ is functorial. Let $\pi : P \to B$ be a principal G-bundle and F a left G-space. The constant map $F \to *$ is G-equivariant, hence induces $P \times_G F \to P \times_G * = B$. This is a fibre bundle w/ fiber F and structure group G.

Exercise Any fibre bundle $F \to B$ w/ structure group G can be reconstructed from a principal bundle using this operation.

Exercise This operation commutes w/ the pullback: that is, if $f : B' \to B$ then there is a natural homeomorphism $f^*(P \times_G F) \cong (f^*P) \times_G F$.

Next Time Relationship between vector bundles & principle bundles, & classifying spaces.