Example For the regular representation $R = \mathbb{k}[G]$, $\chi_R(g) = 0$ for $g \neq 1$ and $\chi_R(1) = |G|$.

Lemma If $\mathbb{k} = \mathbb{C}$ and G finite, for any finite dim'l ρ and $g \in G$ we have $\chi_\rho(g) = \overline{\chi_\rho(g^{-1})}$.

Pf We have $\chi_\rho(g)$ is the sum of the eigenvalues of ρ_g. Since g has finite order, every eigenvalue of ρ_g is a root of 1. The eigenvalues of $\rho_{g^{-1}} = \rho_g^{-1}$ are the complex conjugates.

Orthogonality Let G be finite, $\mathbb{k} = \mathbb{C}$.

We put a nondegenerate symmetric bilinear form on $\mathcal{F}(G)$ via

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) \psi(g)$$

If $\rho: G \to GL(V)$ is a representation, let V^G be the subspace of G-invt vectors, i.e. $V^G = \{v : \rho_g(v) = v \ \forall g \in G\}$

Lemma If $\rho: G \to GL(V)$ is a representation, $\dim V^G = \langle \chi_\rho, \chi_{triv} \rangle$.

PF We have $P \in \mathrm{End}_G(V)$ given by $P = \frac{1}{|G|} \sum_{g \in G} \rho_g$. Note $P^2 = P$, $\mathrm{Im}\, P = V^G$. So P projects onto V^G. Since $\mathrm{char}\, \mathbb{C} = 0$, $\mathrm{tr}\, P = \dim \mathrm{Im}\, P = \dim V^G$. But also $\mathrm{tr}\, P = \langle \chi_\rho, \chi_{triv} \rangle$ by computation.

Corollary $\dim \mathrm{Hom}_G(V, W) = \langle \chi_\rho, \chi_\sigma \rangle$

PF $\langle \chi_\rho, \chi_\sigma \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g^{-1}) \chi_\sigma(g) = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho^* \otimes \sigma}(g) = \langle \chi_{\rho^* \otimes \sigma}, \chi_{triv} \rangle$

Since $\mathrm{Hom}_G(V, W) = (V^* \otimes W)^G$ we are now done.

Thm Let ρ, σ be irreducible representations.

(a) If $\rho: G \to GL(V)$ and $\sigma: G \to GL(W)$ are not isomorphic, $\langle \chi_\rho, \chi_\sigma \rangle = 0$.

(b) If ρ and σ are equivalent, $\langle \chi_\rho, \chi_\sigma \rangle = 1$.

PF (a) Schur's Lemma $\Rightarrow \mathrm{Hom}_G(V, W) = 0$.

(b) $\langle \chi_\rho, \chi_\sigma \rangle = \dim \mathrm{Hom}_G(V, W) = \dim(\mathbb{C}) = 1$.

Corollary Let $\rho = m_1\rho_1 \oplus \cdots \oplus m_r\rho_r$ be a decomposition into a sum of irreducible representations, where $m_i\rho_i$ is the direct sum of m_i copies of ρ_i. Then $m_i = \dfrac{\langle \chi_\rho, \chi_{\rho_i} \rangle}{\langle \chi_{\rho_i}, \chi_{\rho_i} \rangle}$. We say $V_i^{\oplus m_i}$ are the isotypic components of ρ.

Corollary Two finite-dim'l representations ρ and σ are equivalent $\Leftrightarrow$ their characters coincide.

Corollary A representation ρ is irreducible $\Leftrightarrow \langle \chi_\rho, \chi_\rho \rangle = 1$.

Thm Every irreducible representation ρ of G appears in the regular representation w/ multiplicity $\dim \rho$.

PF $\langle \chi_\rho, \chi_R \rangle = \frac{1}{|G|} \chi_\rho(1) \chi_R(1) = \dim \rho$.

Corollary Let $\rho_1, \ldots, \rho_r$ be the irreducible representations of G and $n_i = \dim \rho_i$. Then $n_1^2 + \cdots + n_r^2 = |G|$.

PF $\dim R = |G| = \chi_R(1) = \sum_{i=1}^r n_i \chi_{\rho_i}(1) = \sum_{i=1}^r n_i^2$.

The number of representations of a finite group

Defn Let $c(G) = \{ \varphi \in F(G) : \varphi(ghg^{-1}) = \varphi(h) \}$ be the class-functions.

Exercise $\langle , \rangle$ restricts to a nondegenerate form on $c(G)$.

Thm The characters of the irreducible representations on G form an orthonormal basis of $c(G)$.

Pf Wts that if $\varphi \in c(G)$ and $\langle \varphi, \chi_\rho \rangle = 0$ for any irreducible representation ρ, then $\varphi = 0$.

Claim Let $\rho : G \to GL(V)$ be a representation, $\varphi \in c(G)$ and

$$T = \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) \rho_g$$

Then $T \in \mathrm{End}_G V$ and $\mathrm{tr}\, T = \langle \varphi, \chi_\rho \rangle$.

This is an exercise. Then for ρ irreducible we have that

$$\frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) \rho_g = 0$$

But any representation is a direct sum of irreducibles, so this is true of <u>any</u> representation. In particular for the regular representation R we have

$$\frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) R_g(1) = \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) g = 0$$

Hence $\varphi(g^{-1}) = 0$ for all $g \in G$, i.e. $\varphi = 0$. $\square$

<u>Corollary</u> The number of isomorphism classes of irreducible representations is the number of conjugacy reps of G.

<u>Corollary</u> If G is a finite abelian group, every irreducible representation is one-dimensional and the number of irreducible representations is $|G|$.

<u>Remark</u> If G^* is the set of one-dim'l irreducible representations of G, we have

- G^* is a group wrt $\otimes$
- $G^* \simeq (G/[G,G])^*$ since

$$\begin{tikzcd} p : G \arrow[r] \arrow[dr] & GL(\mathbb{C}) = \mathbb{C}^* \\ & G/[G,G] \arrow[u] \end{tikzcd}$$

- If G is finite abelian, $G \simeq G^*$, noncanonically.

<u>Example</u> $S_n/[S_n, S_n] = S_n/A_n = \{\pm 1\} \longrightarrow \mathbb{C}^*$ has two maps to $\mathbb{C}^*$, the trivial rep and the sign rep.

Example S_3 over $\mathbb{C}$

		1	3	2
		(1)	(12)	(123)
V_1	χ_1	1	1	1
V_2	χ_2	1	-1	1
V_3	χ_3	2	0	-1

We can use

$$\chi_{perm} = \chi_1 + \chi_3$$

$W = \{k(1,1,1) : k \in \mathbb{C}\}$ $\quad W' = \{(x,y,z) : x+y+z=0\}$

$\chi_{perm}((1)) = 3$

$\chi_{perm}((12)) = 1$

$\chi_{perm}((123)) = 0$

<u>Exercise</u> For V an irrep, $V \otimes V$ contains $V_0 \iff V$ is self dual.

$V_2 \otimes V_2 = V_1$

$V_2 \otimes V_3 = V_3$] Tensoring w/ a 1-dim'l irrep is invertible

$V_3 \otimes V_3 = V_1 \oplus V_2 \oplus V_3 \quad \leftarrow \boxed{4 \mid 0 \mid 1}$

S_4 over $\mathbb{C}$

	1	6	8	3	6
	(1)	(12)	(123)	(12)(34)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	-1	1
χ_5	2	0	-1	2	0
		a	b	c	d

- $\chi_{perm} = \chi_1 + \chi_3$
- $\rho_2 \otimes \rho_3$ is also irreducible
- χ_5 can now be produced from orthogonality.

$\left.\begin{array}{l} \cdot\ 2 + 6a + 8b + 3c + 6d = 0 \\ \cdot\ 2 - 6a + 8b + 3c - 6d = 0 \end{array}\right\} \quad \begin{array}{l} 8b + 3c + 2 = 0 \\ a + d = 0 \end{array}$

$\left.\begin{array}{l} \cdot\ 6 + 6a - 3c - 6d = 0 \\ \cdot\ 6 - 6a - 3c + 6d = 0 \end{array}\right\} \quad \begin{array}{l} 6 = 3c \\ a - d = 0 \end{array}$

We can consider the ring $\mathrm{Rep}(G)$ w/ basis $[V_1], \dots, [V_n]$ the irreps over $\mathbb{C}$ w/ $[V_i] + [V_j] = [V_i \oplus V_j]$ and $[V_i] \times [V_j] = [V_i \otimes V_j]$

We have structure constants a_{ij}^k st $V_i \otimes V_j = \bigoplus V_k^{a_{ij}^k}$ given by $\chi_i \chi_j = \sum a_{ij}^k \chi_k$.

Module rephrasing

- Recall a representation ρ of G is equivalently a module over $\Bbbk[G]$.
- An irreducible representation is a simple module (no nonzero proper submodules). A representation that can be decomposed into a direct sum of irreducibles corresponds to a semisimple module (can be decomposed into a direct sum of simples).
- $\Bbbk[G]$ is a semisimple ring (= "semisimple as a module over itself") $\Leftrightarrow$ every representation of G over $\Bbbk$ decomposes into a direct sum of irreducibles.
  - eg $\mathbb{Q}[G]$ semisimple for G finite

In this formulation Schur's Lemma Let M and N be simple R-modules. If $\mathrm{Hom}_R(M,N)$ is not zero then it is an isomorphism. If M is a simple module, then $\mathrm{End}_R(M)$ is a division ring.

The structure of the group algebra: product of matrix rings

Recall Given a ring R, one has the ring R^{op} w/ the same abelian group structure and multiplication $a * b = ba$.

Lemma The ring $\mathrm{End}_R(R)$ is isomorphic to R^{op}.

Pf Given $a \in R$, we let $\varrho_a(x) = xa$. Then $\varrho_a \in \mathrm{End}_R(R)$ and $\varrho_{ba} = \varrho_a \circ \varrho_b$. This constructs a homomorphism $\varrho : R^{op} \to \mathrm{End}(R)$. It is straightforward to check it is an isomorphism.

<u>Lemma</u> Let $\rho_i : G \to GL(V_i)$, $i = 1, \dots, l$ be distint irreducible representations of a finite group over an algebraically closed field. Let $V = V_1^{\oplus m_1} \oplus \dots \oplus V_l^{\oplus m_l}$

Then $\mathrm{End}_G(V) \cong M_{m_1}(\mathbb{k}) \times \dots \times M_{m_l}(\mathbb{k})$, where $M_n(\mathbb{k})$ denotes the ring of $n \times n$ matrices.

<u>PF</u> If $\varphi \in \mathrm{End}_G(V)$, Schur's Lemma implies that φ preserves isotypic components So we have an isomorphism $\mathrm{End}_G(V) \cong \mathrm{End}_G(V_1^{\oplus m_1}) \times \dots \times \mathrm{End}_G(V_l^{\oplus m_l})$.

So it suffices to check that for W a simple $\mathbb{k}[G]$-module, $\mathrm{End}_G(W^{\oplus m})$ is isomorphic to $M_m(\mathbb{k})$.

<u>PF</u> For $i, j = 1, \dots, m$ let p_j be the projection of $W^{\oplus m}$ onto its jth factor, q_i the inclusion map on the ith factor. Let $\varphi \in \mathrm{End}_G(W^{\oplus m})$. Let φ_{ij} be the composition $W \xrightarrow{q_i} W^{\oplus m} \xrightarrow{\varphi} W^{\oplus m} \xrightarrow{p_i} W$. By Schur's Lemma, $\varphi_{ij} = c_{ij} \, \mathrm{Id}_W$ for some $c_{ij} \in \mathbb{k}$. Hence there is a map $\Phi : \mathrm{End}(W^{\oplus m}) \to M_m(\mathbb{k})$, plainly injective & surjective by construction. For homomorphism, note we can write $\varphi = \sum_{i,j=1}^{m} c_{ij} \, q_i \circ p_j$. If $\psi = \sum_{i,j=1}^{m} d_{ij} \, q_i \circ p_j$, we see that $\varphi \circ \psi = \sum_{i,j,k=1}^{m} c_{ik} d_{kj} q_i \circ p_j$. $\square$

<u>Propn</u> If G finite and $\mathbb{k}$ algebraically closed, $\mathbb{k}[G] \cong M_{n_1}(\mathbb{k}) \times \dots \times M_{n_r}(\mathbb{k})$ where $n_1, \dots, n_r$ are the dimensions of the distinct irreducible representations.

<u>PF</u> $\mathrm{End}(\mathbb{k}[G]) \cong \mathbb{k}[G]^{op}$. But $g \to g^{-1}$ induces an isomorphism $\mathbb{k}[G]^{op} \cong \mathbb{k}[G]$ so $\mathbb{k}[G] \cong \mathrm{End}(\mathbb{k}[G]) \cong \mathrm{End}(V_1^{\oplus n_1} \oplus \dots \oplus V_r^{\oplus n_r}) \cong M_{n_1}(\mathbb{k}) \times \dots \times M_{n_r}(\mathbb{k})$.