Example For the regular representation \( R \coloneqq H[\mathfrak{g}] \), \( x_R(g) = 0 \) for \( g \neq 1 \) and \( x_R(1) = |\mathfrak{g}| \).

**Lemma** If \( \mathfrak{h} = \mathfrak{g} \) and \( \mathfrak{g} \) finite, for any finite dim \( \mathfrak{h} \) and \( g \in \mathfrak{g} \) we have \( x_{\mathfrak{h}}(g) = x_{\mathfrak{h}}(g^{-1}) \).

**Proof** We have \( x_{\mathfrak{h}}(g) \) is the sum of the eigenvalues of \( g \). Since \( g \) has finite order, every eigenvalue of \( g \) is a root of \( 1 \). The eigenvalues of \( g^{1}, g^{-1} \) are the complex conjugates.

**Orthogonality** Let \( \mathfrak{g} \) be finite, \( \mathfrak{h} = \mathfrak{g} \).

We put a nondegenerate symmetric bilinear form on \( F(\mathfrak{g}) \) via

\[
\langle e, \psi \rangle = \frac{1}{|\mathfrak{g}|} \sum_{g \in \mathfrak{g}} \epsilon(g^{-1}) \psi(g)
\]

If \( \rho: \mathfrak{g} \to \mathfrak{gl}(V) \) is a representation, let \( V^\mathfrak{g} \) be the subspace of \( \mathfrak{g} \)-invariant vectors, i.e. \( V^\mathfrak{g} = \{ v : \rho(g)v = v \ \forall g \in \mathfrak{g} \} \).

**Lemma** If \( \rho: \mathfrak{g} \to \mathfrak{gl}(V) \) is a representation, \( \text{dim } V^\mathfrak{g} = \langle x_{\mathfrak{h}}, x_{\mathfrak{eIV}} \rangle \).

**Proof** We have \( P \in \text{End}_\mathfrak{g}(V) \) given by \( P = \frac{1}{|\mathfrak{g}|} \sum_{g \in \mathfrak{g}} \rho(g) \). Note \( P^2 = P \), \( \text{Im } P = V^\mathfrak{g} \).

So \( P \) projects onto \( V^\mathfrak{g} \). Since \( \text{char } \mathfrak{g} = 0 \), \( \text{tr } P = \text{dim } \text{Im } P = \text{dim } V^\mathfrak{g} \), but also \( \text{tr } P = \langle x_{\mathfrak{h}}, x_{\mathfrak{eIV}} \rangle \) by computation.

**Corollary** \( \text{dim } \text{Hom}_\mathfrak{g}(V, W) = \langle x_{\mathfrak{h}}, x_{\mathfrak{e}} \rangle \)

**Proof** \( \langle x_{\mathfrak{h}}, x_{\mathfrak{e}} \rangle = \frac{1}{|\mathfrak{g}|} \sum_{g \in \mathfrak{g}} x_p(g^{-1}) x_{\mathfrak{e}}(g) = \frac{1}{|\mathfrak{g}|} \sum_{g \in \mathfrak{g}} x_{\mathfrak{h} \mathfrak{e}}(g) = \langle x_{\mathfrak{h} \mathfrak{e}}, x_{\mathfrak{eIV}} \rangle \)

Since \( \text{Hom}_\mathfrak{g}(V, W) = (V^* \otimes W)^\mathfrak{g} \) we are now done.
Thm. Let $\rho, \sigma$ be irreducible representations.

(a) If $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ are not isomorphic, $\langle \chi_\rho, \chi_\sigma \rangle = 0$.
(b) If $\rho$ and $\sigma$ are equivalent, $\langle \chi_\rho, \chi_\sigma \rangle = 1$.

Proof. Schur's Lemma $\Rightarrow \text{Hom}_G(V, W) = 0$.

(i) $\langle \chi_\rho, \chi_\sigma \rangle = \dim \text{Hom}_G(V, W) = \dim(\sigma) = 1$.

Corollary. Let $\rho = \rho_1 \oplus \cdots \oplus \rho_r$ be a decomposition into a sum of irreducible representations, where $\rho_i \cong \rho_i$ is the direct sum of $m_i$ copies of $\rho_i$. Then $m_i = \frac{\langle \chi_{\rho_i}, \chi_{\rho_i} \rangle}{\langle \chi_\rho, \chi_\rho \rangle}$. We say $V_i \cong \rho_i$ are the isotypic components of $\rho$.

Corollary. Two finite-dimensional representations $\rho$ and $\sigma$ are equivalent if their characters coincide.

Corollary. A representation $\rho$ is irreducible $\iff \langle \chi_\rho, \chi_\rho \rangle = 1$.

Thm. Every irreducible representation $\rho$ of $G$ appears in the regular representation with multiplicity $\dim \rho$.

Proof. $\langle \chi_\rho, \chi_R \rangle = \frac{1}{|G|} \chi_\rho(1) \chi_R(1) = \dim \rho$.

Corollary. Let $\rho_1, \ldots, \rho_r$ be the irreducible representations of $G$ and $n_i = \dim \rho_i$.

Then $n_1^2 + \cdots + n_r^2 = |G|.

Proof. $\dim R = |G| = \chi_R(1) = \sum_{i=1}^r n_i \chi_{\rho_i}(1) = \sum_{i=1}^r n_i^2$. 
The number of representations of a finite group

**Def.** Let \( C(G) = \{ \varphi \in \text{F}(G) : \varphi(gh^{-1}) = \varphi(h)^2 \} \) be the class-functions.

**Exercise.** \(<,>\) restricts to a nondegenerate form on \( C(G) \).

**Thm.** The characters of the irreducible representations on \( G \) form an orthonormal basis of \( C(G) \).

**PF.** We say that if \( \varphi \in C(G) \) and \( \langle \varphi, \chi_p \rangle = 0 \) for any irreducible representation \( p \), then \( \varphi = 0 \).

**Claim.** Let \( p : G \to GL(V) \) be a representation, \( \varphi \in C(G) \) and

\[
T = \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) p_g
\]

Then \( T \in \text{End}_G V \) and \( \text{tr} T = \langle \varphi, \chi_p \rangle \).

This is an exercise. Then for \( p \) irreducible we have that

\[
\frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) p_g = 0
\]

But any representation is a direct sum of irreducibles, so this is true of any representation. In particular for the regular representation \( R \) we have

\[
\frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) R_g(1) = \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) = 0
\]

Hence \( \varphi(g^{-1}) = 0 \) for all \( g \in G \), i.e. \( \varphi = 0 \). \( \Box \)
Corollary 1. The number of isomorphism classes of irreducible representations is the number of conjugacy reps of $G$.

Corollary 2. If $G$ is a finite abelian group, every irreducible representation is one-dimensional and the number of irreducible representations is $|G|$.

Remark 1. If $G^*$ is the set of one-dim'l irreducible representations of $G$, we have

- $G^*$ is a group under $\otimes$
- $G^* \cong (G/[G,G])^*$ since $\rho: G \to GL(C) = C^*$

Example: $S_n /[S_n, S_n] / A_n \to C^*$ has two maps to $C^*$, the trivial rep and the sign rep.
Example $S_3$ over $C$

\[
\begin{array}{ccc}
V_1 & 1 & 3 & 2 \\
\chi_1 & 1 & 1 & 1 \\
\chi_2 & 1 & -1 & 1 \\
\chi_3 & 2 & 0 & -1 \\
\end{array}
\]

Exercise. For $V$ an irrep, $V \otimes V$ contains $V_0 \Rightarrow V$ is self-dual.

$S_4$ over $C$

\[
\begin{array}{cccccc}
1 & 3 & 5 & 6 & 8 & 9 \\
(1) & (12) & (123) & (1234) & (1234) & (1234) \\
\chi_1 & 1 & 1 & 1 & 1 & 1 \\
\chi_2 & 1 & 1 & 1 & -1 & -1 \\
\chi_3 & 3 & 0 & 0 & -1 & -1 \\
\chi_4 & 3 & -1 & 0 & -1 & 1 \\
\chi_5 & 2 & 0 & -1 & 2 & 0 \\
\end{array}
\]

We can use

\[x_{\text{perm}} = x_1 + x_3\]

\[w = \frac{1}{6} \sum_{i,j} \chi_i(x) \chi_j(x) \Rightarrow x + yz = 0\]

\[x_{\text{perm}}((1)) = 3\]

\[x_{\text{perm}}((12)) = 1\]

\[x_{\text{perm}}((123)) = 0\]

\[V_1 \otimes V_2 = V_1\]

\[V_2 \otimes V_3 = V_3\] Tensoring w/ a 1-dim'l irrep is invertible.

\[V_3 \otimes V_3 = V_1 \otimes V_2 \otimes V_3 \triangleleft 4 \oplus 1 (1)\]

\[x_{\text{perm}} = x_1 + x_3\]

\[P_2 \otimes P_3\] is also irreducible.

\[x_{\text{S}}\] can now be produced from orthogonality.

\[
\begin{align*}
2 + 6a + 8b + 3c + 4d &= 0 \\
8b + 3c + 2d &= 0 \\
2 - 6a + 8b + 3c - 6d &= 0 \\
a + d &= 0 \\
6 + 6a - 3c - 6d &= 0 \\
6 - 6a - 3c + 6d &= 0 \\
a - d &= 0
\end{align*}
\]

We can consider the ring $\text{Rep}(G)$ w/ basis $[V_1], \ldots, [V_6]$.

The irreps over $C$ w/ $[V_i] \ast [V_j] = [V_i \otimes V_j]$ and $[V_i] \ast [V_j] = [V_i \otimes V_j]$.

We have structure constants $a_{ij}^k$ s.t. $V_i \otimes V_j = \bigoplus V_k a_{ij}^k$ given by $x_i x_j = \sum a_{ij}^k x_k$.
Module rephrasing

* Recall a representation \( \rho \) of \( G \) is equivalently a module over \( \mathbb{K}[G] \).

  * An irreducible representation is a simple module (no nonzero proper submodules). A representation that can be decomposed into a direct sum of irreducibles corresponds to a semisimple module (can be decomposed into a direct sum of simples).

  * \( \mathbb{K}[G] \) is a semisimple ring (= "semisimple as a module over itself") \( \implies \) every representation of \( G \) over \( \mathbb{K} \) decomposes into a direct sum of irreducibles.

  \( \implies \) \( \mathbb{K}[G] \) semisimple for \( G \) finite.

In this formulation **Schur's Lemma** let \( M \) and \( N \) be simple \( R \)-modules. If \( \text{Hom}_R(M,N) \) is not zero then it is an isomorphism. If \( M \) is a simple module, then \( \text{End}_R(M) \) is a division ring.

The structure of the group algebra: product of matrix rings

* Recall given a ring \( R \), one has the ring \( R^\text{op} \) w/ the same abelian group structure and multiplication \( a \circ b = ba \).

**Lemma** The ring \( \text{End}_R(R) \) is isomorphic to \( R^\text{op} \).

**PF** Given \( a \in R \), we let \( E_a(x) = xa \). Then \( E_a \in \text{End}_R(R) \) and \( E_a \circ E_b = E_a \circ E_b \). This constructs a homomorphism \( \Phi: R^\text{op} \rightarrow \text{End}_R(R) \). It is straightforward to check it is an isomorphism.
Lemma Let $p_i : G \to GL(V)$, $i = 1, \ldots, r$ be distinct irreducible representations of a finite group over an algebraically closed field. Let $V = V_{\otimes m_1} \oplus \cdots \oplus V_{\otimes m_r}$.

Then $End_G(V) \cong M_{m_1}(k) \times \cdots \times M_{m_r}(k)$, where $M_n(k)$ denotes the ring of $n \times n$ matrices.

Proof If $\phi \in End_G(V)$, Schur's Lemma implies that $\phi$ preserves isotypic components. So we have an isomorphism $End_G(V) \cong End_G(V_{\otimes m_1}) \times \cdots \times End_G(V_{\otimes m_r})$.

So it suffices to check that for $W$ a simple $k[G]$-module, $End_G(W^{\otimes m})$ is isomorphic to $M_{m_1}(k)$.

Proof For $i = 1, \ldots, m$ let $p_i$ be the projection of $W^{\otimes m}$ onto its $i$th factor, $\phi_i$ the inclusion map on the $i$th factor. Let $\phi \in End_G(W^{\otimes m})$. Let $\phi_i$ be the composition $W \xrightarrow{\phi_i} W^{\otimes m} \xrightarrow{\phi_i} W^{\otimes m} \xrightarrow{\phi_i} W$. By Schur's Lemma, $\phi_i = c_{ij} \text{Id}_W$ for some $c_{ij} \in k$. Hence there is a map $\tilde{\phi} : End(W^{\otimes m}) \to M_{m_1}(k)$, plainly injective and surjective by construction. For homomorphism, note we can write $\phi = \sum_{i,j} c_{ij} \phi_i$ so $\phi = \sum_{i,j} c_{ij} \phi_i$. Thus we see that $\phi \circ \psi = \sum_{i,j} c_{ij} \phi_i \circ \psi_i$.

Prop If $G$ finite and $k$ algebraically closed, $k[G] \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k)$ where $n_1, \ldots, n_r$ are the dimensions of the distinct irreducible representations.

Proof $End(k[G]) \cong k[G]^\text{op}$. But $g \mapsto g^{-1}$ induces an isomorphism $k[G]^\text{op} \cong k[G]$.

$\therefore k[G] \cong End(k[G]) \cong End(V_{\otimes n_1} \oplus \cdots \oplus V_{\otimes n_r}) \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k)$.