Note: It is helpful to know the answers to these two exercises for the following lecture.

Math 549: Suggested Exercises for Lectures 18 and 19

Bump Chapter 12, Kirillov Sections 4.8-9

1. Let $V_k = S^k\mathbb{C}^2$ be the representation of $\mathfrak{sl}(2,\mathbb{C})$ constructed in class.
   (a) Show that V_2 is isomorphic to the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$.
   (b) Which of the representations V_k lift to representations of $SO(3,\mathbb{R})$?
2. Show that $\Lambda^n\mathbb{C}^n \simeq \mathbb{C}$ as a representation of $\mathfrak{sl}(n,\mathbb{C})$. Does this also work for $\mathfrak{gl}(n,\mathbb{C})$.
3. If (ρ, V) is a representation of $SL(2,\mathbb{R})$, $SU(2)$ or $SL(2,\mathbb{C})$, we can restrict the character function of ρ to the diagonal subgroup, obtaining a Laurent polynomial

$$\xi_\rho = \operatorname{tr}\rho \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

   (a) Compute $\xi_\rho(t)$ for the symmetric power representations. Show that the resulting polynomials are independent and determine the representation ρ.
   (b) Show that $\xi_{\rho\otimes\rho'} = \xi_\rho \xi\rho'$. Use this to determine the decomposition of products of the symmetric power representations into irreducibles.
4. Let V be a representation of $(2,\mathbb{C})$, and let $C \in \operatorname{End}(V)$ such that $C(v)$ is

$$EFv - FEv + \frac{1}{2}H^2 v.$$

   (a) Show that C commutes with the action of $(2,\mathbb{C})$, that is, show that for $x \in \mathfrak{sl}(2,\mathbb{C})$, then $[\rho(x), C] = 0$. (Hint: Jacobi Identity.)
   (b) Show that if $V = V_k$ is the irreducible representation with highest weight k, then $C = c_k\mathrm{Id}$. Compute the constant c_k.
   (c) Show that the isomorphism $\mathfrak{so}(3,\mathbb{C}) \simeq \mathfrak{sl}(2,\mathbb{C})$ identifies C with a multiple of $\rho(J_x)^2 + \rho(J_y)^2 + \rho(J_z)^2$.

   This is a special case of the *Casimir element*, which we'll discuss more shortly.
5. Complete the computation from lecture (or Kirillov Section 4.9) to find the eigenvalues and multiplicities of the operator

This one is for afterword

$$H = -\Delta - \frac{c}{r}, \qquad c > 0$$

in $L^2(\mathbb{R}^3, \mathbb{C})$.

Lecture 19: The Laplace operator on the sphere

Ref Kirillov 4.9 (and Introduction)

Let $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ be the Laplace operator on $\mathbb{R}^3$.

We split it into radial and spherical parts using the decomposition $\mathbb{R}^3 - \{0\} \simeq S^2 \times \mathbb{R}_+$ $\quad \vec{u} = \frac{\vec{x}}{|\vec{x}|} \in S^2, \; r = |\vec{x}| \in \mathbb{R}_+$

$$x \longmapsto (\vec{u}, r)$$

Lemma In coordinates $(\vec{u}, r)$, $\Delta|_{\mathbb{R}^3 - \{0\}}$ decomposes as

$$\Delta = \frac{1}{r^2}\Delta_{sph} + \Delta_{radial}$$

where $\Delta_{radial} = \partial_r^2 + \frac{2}{r}\partial_r$ and $\Delta_{sph} = J_x^2 + J_y^2 + J_z^2$ where

$$J_x = y\partial_z - z\partial_y$$

$$J_y = z\partial_x - x\partial_z$$

$$J_z = x\partial_y - y\partial_x$$

Pf This is just multivariable calculus.

Notice that these are exactly the left-invt vector fields corresponding to

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad J_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

which generate $\mathfrak{so}(3,\mathbb{R})$ via rotations around the x, y, and z axes.
So we see the action of $\mathfrak{so}(3,\mathbb{R})$ on $C^\infty(S^2, \mathbb{C}^3)$ very directly.

We can write Δ_{sph} in terms of the usual coordinates for the sphere, but this is extremely messy.

Problem Find the eigenvalues of Δ_{sph} acting on $C^\infty(S^2)$.

Why important? This is physics-motivated.

A particle moving in a central force field (eg an electron in the hydrogen atom) is described by the Schrödinger equation

$$\dot{\psi} = iH\psi$$

where $\psi = \psi(t, \vec{x})$, $\vec{x} \in \mathbb{R}^3$ is a wave function describing the state of the system, the dot is the time derivative, and

$$H = -\Delta + V(r)$$

is the Hamiltonial. $V(r)$ is a potential function describing the central force field. Solving the equation $\Leftrightarrow$ diagonalizing the Hamiltonian. The straightforward approach involves writing this out in coordinates & using separation of variables - very explicit & maybe messy.

Lie groups solution

Lemma $\Delta_{sph} : C^\infty(S^2) \longrightarrow C^\infty(S^2)$ commutes w/ the action of $SO(3, \mathbb{R})$.

PF It's pretty clear both that Δ and Δ_{radial} are rotation-invariant. Alternately, recall that $\Delta_{sph} = J_x^2 + J_y^2 + J_z^2$. We claim that for any representation (ρ, V) of $so(3)$, $C = \rho(J_x)^2 + \rho(J_y)^2 + \rho(J_z)^2$

commutes w/ the action of $SO(3,\mathbb{R})$. Equivalently we claim that for any $a \in \mathfrak{g}$, $[\rho(a), c] = 0$. In general this follows from the theory of Casimir elements in the universal enveloping algebra, but in the specific we observe that $J_x^2 + J_y^2 + J_z^2 = -2\,Id$. $\square$

So in order to understand Δ_{sph}, it is desirable to decompose the space of functions on S^2 into irreducible representations of $SO(3,\mathbb{R})$. We make these complex-valued functions since we understand cpx representations.

__DeFn__ Let P_n be the space of complex-valued functions on S^2 which can be written as polynomials in x, y, z of total degree $\leq n$.

Each P_n is a finite-dim'l representation of $SO(3)$ which is Δ_{sph}-invariant. Hence, we should be able to decompose P_n into irreps and use this to find eigenvalues of Δ_{sph} in P_n.

__Recall__ (From last time and hw) Irreps of $SO(3,\mathbb{R})$ are of the form V_{2k}, $k \in \mathbb{Z}_+$. Hence $P_n = \bigoplus c_k V_{2k}$ for some coefficients c_k.

To find the coefficients c_k, we need to find the dimensions of the weight spaces, which are the eigenspaces of J_z.

<u>Lemma</u> The following functions form a basis for P_n:

$$f_{p,k} = z^p \left(\sqrt{1-z^2}\right)^{|k|} e^{ik\varphi}$$

For $p \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, $p+|k| \leq n$ and φ is defined by $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $\rho = \sqrt{x^2+y^2}$.

<u>PF</u> Let $u = x+iy = \rho e^{i\varphi}$, $v = x-iy = \rho e^{-i\varphi}$. Then we can write any polynomial in x, y, z in terms of z, u, v. On the sphere we have $1-z^2 = x^2+y^2 = uv$, so in fact every monomial $z^k u^\ell v^m$ can be rewritten as a term involving only one of u or v. Ie, we can write monomials

- z^p
- $z^p u^k = z^p \rho^k e^{ik\varphi} = f_{p,k}$
- $z^p v^k = z^p \rho^k e^{-ik\varphi} = f_{p,-k}$

w/ $p, k \in \mathbb{Z}_+$, $p+k \leq n$. So the functions $f_{p,k}$ span P_n. For linear independency suppose $0 = \sum a_{p,k} f_{p,k} = \sum_k a_k(z) e^{ik\varphi} = 0$, $a_k(z) = \sum_p a_{p,k} z^p \left(\sqrt{1-z^2}\right)^{|k|}$.

By uniqueness of Fourier series, we see that for every $k \in \mathbb{Z}$ and $z \in (-1,1)$, we have that $a_k(z) = 0$. This implies that for each pair p, k, $a_{p,k} = 0$. $\square$

Now since J_z generates rotations around the z-axis, we see that in coordinates z, ρ, φ, we have $J_z = \frac{\partial}{\partial\varphi}$. Therefore $J_z \cdot f_{p,k} = ik \cdot f_{p,k}$; that is, we've written down a basis of eigenvectors for P_n.

We see that $P_n[2k] = \mathrm{Span}(f_{p,k})_{0 \le p \le n-|k|}$, so $\dim P_n[2k] = n+1-k$.

We see that $P_n \simeq V_0 \oplus V_2 \oplus \cdots \oplus V_{2n}$.

Now we can compute the eigenvalues of $\Delta = J_x^2 + J_y^2 + J_z^2$ easily; by homework exercise, $J_x^2 + J_y^2 + J_z^2$ acts in V_ℓ by $\frac{-\ell(\ell+2)}{4}$.

Ergo

Thm The eigenvalues of the spherical Laplace operator Δ_{sph} in the space P_n are $\lambda_k = -k(k+1)$, $k = 0, \ldots, n$, and the multiplicity of λ_k is equal to $\dim V_{2k} = 2k+1$.

Thm Each eigenfunction of Δ_{sph} is polynomial. The eigenvalues are $\lambda_k = -k(k+1)$, $k \in \mathbb{N}$, and the multiplicity of λ_k is $2k+1$.

Pf Consider the space $L^2(S^2, \mathbb{C})$ of complex-valued L^2-functions on S^2. The action of $SO(3)$ preserves the volume form on S^2, and therefore also preserves the inner product on $L^2(S^2, \mathbb{C})$. Hence the operators J_x, J_y, J_z are skew-Hermitian (since the operators they exponentiate to are Hermitian). In particular $\Delta = J_x^2 + J_y^2 + J_z^2$ is Hermitian (the square of a skew-Hermitian operator is skew-Hermitian) and in particular self-adjoint.

Let $E_n \subseteq P_n$ be the orthogonal complement to P_{n-1}. Then E_n is $SO(3)$-invt, and from the decomposition of P_n and P_{n-1} into irreducibles, we conclude that $E_n \simeq V_{2n}$.

So Δ_{sph} acts on E_n by λ_n. On the other hand, the space of polynomials is dense in L^2, so we have that

$$L^2(S^2, \mathbb{C}) = \bigoplus_{n \geq 0} E_n$$

as a sum of Hilbert spaces. So if $\Delta_{sph} f = \lambda f$ for some function $f \in C^\infty(S^2) \subseteq L^2(S^2, \mathbb{C})$, then either $\lambda \neq \lambda_n$ for all n, forcing $(f, E_n) = 0$ for all n, or $\lambda = \lambda_n$, so $(f, E_k) = 0$ for all $k \neq n$, implying that $f \in E_n$. $\square$

<u>Exercise</u> Find the eigenvalues and multiplicities of

$$\underbrace{H = -\Delta - \frac{c}{r}}_{\text{hydrogen atom}} \qquad c > 0$$