Example: Representations of $\mathfrak{sl}(2,\mathbb{C})$

Reminder

A representation $\rho: \mathfrak{g} \to GL(V)$ of a complex Lie algebra is assumed complex linear; a representation $\rho: \mathfrak{g} \to GL(V)$ of a real Lie algebra is real linear.

What are some representations of $SL(2,\mathbb{C})$?

- Standard representation on $\mathbb{C}^2$
- Let $S^k V$ be the space of symmetric tensors

$$S^k V = V^{\otimes k} \Big/ \{ v_1 \otimes \cdots \otimes v_k - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \,;\, \sigma \in S_k \}$$

$$= \Big\{ v_1 \vee \cdots \vee v_k : v_1 \vee \cdots \vee v_k = \frac{1}{k!} \sum_{\sigma \in S_k} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \Big\}$$

Remark Representations of $\mathfrak{sl}(2,\mathbb{C}) \leftrightarrow$ representations of $\mathfrak{su}(n) \leftrightarrow$ representations of $SU(n)$, so every finite dim'l complex rep is completely reducible

There is an induced representation of $SL(2,\mathbb{C})$ on $\vee^k \mathbb{C}^2 \simeq \mathbb{C}^{k+1}$.

Note that $k=0 \rightsquigarrow \vee^0 \mathbb{C}^2 = \mathbb{C}$.

Corresponding reps of $\mathfrak{sl}(2,\mathbb{R})$ $(\subset \mathbb{C})$

Recall $\mathfrak{sl}(2,\mathbb{R}) = $

$$\Big\langle \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{E}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{F}, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{H} \Big\rangle$$

$$[H,E] = 2E$$
$$[H,F] = -2F$$
$$[E,F] = H$$

Let $\mathbb{C}^2 = \Big\langle \overset{x}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \overset{y}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \Big\rangle$ and consider a basis for $S^k V$ given by

$$v_{k-2\ell} = \underbrace{x \vee \cdots \vee x}_{k-\ell} \vee \underbrace{y \vee \cdots \vee y}_{\ell}$$ for indices $k, k-2, k-4, \ldots, -k$,

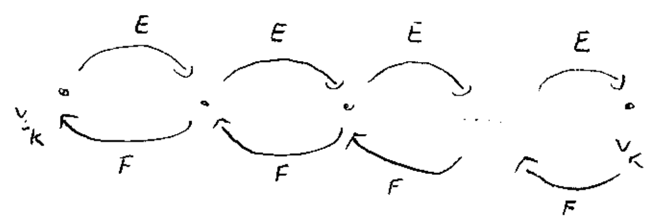
Propn We have

$$H \cdot v_{k-2\ell} = (k-2\ell)\, v_{k-2\ell} \qquad 0 \le \ell \le k$$

$$E \cdot v_{k-2\ell} = \begin{cases} \ell\, v_{k-2\ell+2} & \ell > 0 \\ 0 & \ell = 0 \end{cases}$$

$$F \cdot v_{k-2\ell} = \begin{cases} (k-\ell)\, v_{k-2\ell-2} & \ell < k \\ 0 & \ell = k \end{cases}$$

One visualizes this setup as



Each vector spans an eigenspace of H, called a weight space.

PF For example consider the effect of E on v_i. In $\mathbb{C}^2$, we have the map

$$\exp(tE) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \qquad\qquad E \cdot v = \frac{d}{dt}(\exp tE)_v \Big|_{t=0}$$
$$x \longmapsto x$$
$$y \longmapsto y + tx$$

such that in $S^k V$ we have

$$\exp(tE)(v_{k-2\ell}) = \exp(tE)(\overbrace{x \vee \cdots \vee x}^{k-\ell} \vee \overbrace{y \vee \cdots \vee y}^{\ell})$$
$$= (x \vee \cdots \vee x \vee (y+tx) \vee \cdots \vee (y+tx))$$
$$= v_{k-2\ell} + \ell t (v_{k-2\ell+2}) + \binom{\ell}{2} t^2 (v_{k-2\ell+4}) + \cdots$$

If we differentiate with respect to t and then evaluate at $t=0$, we get $E \cdot v_{k-2\ell} = \ell \cdot v_{k-2\ell+2}$. F and H are quite similar.

Example If $k=3$, $S^3\mathbb{C}^2$ has a basis v_3, v_1, v_{-1}, v_{-3} and the actions of E, F, and H are

$$E \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad F \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad H \mapsto \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

Propn $S^k\mathbb{C}^2 = V_k$ is an irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$.

PF Suppose that U is a nonzero invariant subspace. Choose a nonzero element $\sum a_{k-2\ell} v_{k-2\ell}$, and let $k-2\ell$ be the smallest integer st $a_{k-2\ell} \neq 0$. Applying E ℓ times kills every basis vector v_r with $r \geq k-2\ell$, leaving a nonzero constant multiplied by v_k. Hence $v_k \in U$. But applying F repeatedly shows v_{k-2}, v_{k-4}, etc are all in U. Hence $U = V_k$.

Now we'd like to show that these are the only irreps.

Lemma Let V be any $\mathfrak{sl}(2,\mathbb{R})$-module, and let v be an H-eigenvector with eigenvalue k. Then Ev if nonzero is an H-eigenvector w/ eigenvalue $k+2$, and Fv if nonzero is an H-eigenvector w/ eigenvalue $k-2$.

PF Observe that as matrices in $GL(n,\mathbb{C})$ (or if you prefer, in the universal enveloping algebra) $HE - EH = [H,E] = 2E$. It makes sense to evaluate these expressions on vectors; that is,

HEv always makes sense. And indeed, $HEv = EHv + 2Ev$
$= E(kv) + 2Ev$
$= (k+2)Ev$

Likewise for Fv. $\square$

Propn Let V be a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$. Let $v_k \in V$ be an H-eigenvector w/ eigenvalue k maximal. Then k is a positive integer and v_k is contained in an irreducible subspace of V isomorphic to $S^k\mathbb{C}^2 = V_k$.

Pf By the maximality of k and the previous proposition, $Ev_k = 0$. Now consider the submodule U of V generated by v_k. This in particular contains the vectors $v_{k-2}, v_{k-4}, \ldots$ given by

$$v_{k-2\ell-2} = \frac{1}{k-\ell} F \cdot v_{k-2\ell}$$

$$= \frac{(k-\ell)!}{k!} F^{\ell} \cdot v_k$$

The vectors $v_k, v_{k-2}, v_{k-4}, \ldots$, if nonzero are all eigenvectors of H w/ eigenvalues $k, k-2, k-4$, etc., so they are linearly independent and only finitely many of them are nonzero. Hence we have $v_k, v_{k-2}, \ldots, v_{k-2m}$ w/ $v_{k-2m} \neq 0$ but $F \cdot v_{k-2m} = 0$. Now consider the action of E on these vectors. We have $Ev_k = 0$. Then

$$kv_k = Hv_k = EFv_k - FEv_k = EFv_k = E(k \cdot v_{k-2}) = kEv_{k-2}.$$

Inductively we see that if $Ev_{k-2\ell} = \ell v_{k-2\ell+2}$ for $\ell > m+1$,

$$(k-2\ell)v_{k-2\ell} = H \cdot v_{k-2\ell} = EFv_{k-2\ell} - FEv_{k-2\ell}.$$

$$= E((k-\ell)v_{k-2\ell-2}) - F(\ell v_{k-2\ell+2})$$

$$= (k-\ell)Ev_{k-2\ell-2} - \ell(k-\ell+1)v_{k-2\ell}$$

which implies that $Ev_{k-2\ell-2} = \left(\frac{k-2\ell+\ell(k-\ell)+\ell}{k-\ell}\right)v_{k-2\ell}$

$$= \left(\frac{(1+\ell)(k-\ell)}{k-\ell}\right)v_{k-2\ell}$$

$$= (\ell+1)v_{k-2\ell}$$

Finally for v_{k-2m}, assuming $Ev_{k-2m} = mv_{k-2m+2}$, we have

$$(k-2m)v_{k-2m} = Hv_{k-2m}$$

$$= EFv_{k-2m} - FEv_{k-2m}$$

$$= 0 - F(mv_{k-2m+2})$$

$$= -(k-m+1)(m)v_{k-2m+2}$$

$\Rightarrow k-2m = -km + m^2 - m$

$\Rightarrow k + km - m - m^2 = 0$

$\Rightarrow k(1+m) - m(1+m) = 0$

$\Rightarrow (k-m)(1+m) = 0$

$\Rightarrow k = m$

So k is in particular a positive integer and $\{v_k, v_{k-2}, \ldots, v_{-k}\}$ span V and comprise a copy of the irrep V_k.

Conclusion

Thm Let (ρ, V) be any irreducible complex representation of $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$, or $\mathfrak{sl}(2,\mathbb{C})$. Then ρ is isomorphic to V_k for some k.

Corollary Let V be a finite-dimensional complex representation of $\mathfrak{sl}(2,\mathbb{C})$. Then

1. V admits a weight decomposition with integer weights,

$$V = \bigoplus V[n]$$

Eigenspace of H w/ eigenvalue n.

2. $\dim V[n] = \dim V[-n]$. Moreover, for $n \geq 0$, the maps

$$E^n : V[n] \to V[-n]$$

$$F^n : V[-n] \to V[n]$$

are isomorphisms.