# Lecture 17

Ref Bump Chapter 4, Kirillov 4.7

Let $G$ be a cpt group w/ Haar measure $dh$. Recall that for $F: G \to \mathbb{C}$ we have

$$|F|_\infty = \sup_{g \in G} |F(g)|$$

$$|F|_p = \left(\int_G |F|^p \, dh\right)^{1/p}$$

$L^\infty(G)$ - space of measurable functions for which $|F|_\infty < \infty$.

$L^p(G)$ - space of measurable functions for which $|F|_p < \infty$.

Note that $|F|_1 \le |F|_2 \le |F|_\infty$.

- First inequality $|F_1| = \langle |F|, 1 \rangle \underset{\text{Cauchy-Schwarz}}{\le} |F|_2 \cdot |1|_2 = |F|_2$

- Second inequality $|F|_2 = \left(\int_G |F|^2 dh\right)^{1/2} \le \int_G \sup_{g \in G} |F| \, dh = \sup_{g \in G} |F(g)| \underbrace{\int_G dh}_{1} = \sup_{g \in G} |F(g)|$

Let $(F_1 * F_2)(g) = \int_G F_1(gh^{-1}) F_2(h) \, dh = \int_G F_1(h) F_2(h^{-1}g) \, dh$.

For $\phi \in C(G)$ let $T_\phi$ denote left convolution w/ $\phi$, i.e.

$$(T_\phi F)g = \int_G \phi(gh^{-1}) F(h) \, dh$$

Propn If $\phi \in C(G)$, then $T_\phi$ is a bounded operator on $L^1(G)$. If $F \in L^1(G)$, then $T_\phi F \in L^\infty(G)$ and $|T_\phi F|_\infty \le |\phi|_\infty |F|_1$.

Pf If $F \in L^1(G)$, then

$$|T_\phi F|_\infty = \sup_{g\in G} \left| \int_G \phi(gh^{-1}) F(h)\, dh \right|$$

$$\le \sup_{g \in G} |\phi(g)| \cdot \int_G |F(h)| dh$$

$$= |\phi|_\infty \cdot |F|_1$$

So $|T_\phi F|_\infty \le C \cdot |F|_1 \Rightarrow$ For all of $* \in \{\infty, 1, 2\}$, $|T_\phi F|_* \le C \cdot |F|_*$. So $T$ is a bounded operator in all three metrics.

Propn If $\phi \in C(G)$, then convolution with $\phi$ is a bounded operator $T_\phi$ on $L^2(G)$ and $|T_\phi| \le |\phi|_\infty$. The operator $T_\phi$ is compact, and if $\phi(g^{-1}) = \overline{\phi(g)}$, it is furthermore self-adjoint.

Pf Since $|F|_1 \le |F|_2 \le |F|_\infty$, we have that $L^\infty(G) \subseteq L^2(G) \subseteq L^1(G)$. By the previous propn, $|T_\phi F|_2 \le |T_\phi F|_\infty \le |\phi|_\infty |F|_1 \le |\phi|_\infty |F|_2$, so the operator norm of $T_\phi$ has $|T_\phi| \le |\phi|_\infty$.

To show $T_\phi$ is compact, by linearity it suffices to check that the image of the unit ball in $L^2(G)$ is sequentially compact. Indeed, since the unit ball in $L^2(G)$ is contained in the unit ball in $L^1(G)$, it suffices to check that $B = \{ T_\phi F : F \in L^1(G),\ |F|_1 \le 1 \}$ is sequentially compact in $L^2(G)$. But indeed it suffices to check sequential compactness in $L^\infty(G)$, since a sequence that converges wrt $|\cdot|_\infty$ converges wrt $|\cdot|_2$. By the first propn, since $T_\phi$ is a bounded operator, $\mathcal{B}$ is certainly bounded. We will show it is (in the vocabulary from last lecture) equicontinuous. First, since $\phi$ is continuous and $G$ is compact, $\phi$ is uniformly continuous. So, given $\varepsilon > 0$, we may find a neighborhood $N$ of the identity such that $|\phi(kg) - \phi(g)| < \varepsilon$ for all $g$ when $k \in N$. Ergo if $f \in L^1(G)$ and $|f|_1 \le 1$, for any $g \in G$ we have

$$|(\phi * f)(kg) - (\phi * f)(g)| = \left| \int_G [\phi(kgh^{-1}) - \phi(gh^{-1})] f(h)\, dh \right|$$

$$\le \int_G |\phi(kgh^{-1}) - \phi(gh^{-1})| |f(h)|\, dh$$

$$\le \varepsilon |f|_1$$

$$\le \varepsilon$$

Since all ftns in $\mathcal{B}$ are of the form $\phi * f$, we conclude that $\mathcal{B}$ is equicontinuous. Therefore $\mathcal{B}$ is sequentially compact by the Arzela-Ascoli lemma.

For self-adjointness, if $\phi(g^{-1}) = \overline{\phi(g)}$, then we have

$$\langle T_\phi f_1, f_2 \rangle = \int_G \int_G \phi(gh^{-1}) f_1(h) \overline{f_2(g)}\, dg\, dh$$

and $$\langle f_1, T_\phi f_2 \rangle = \int_G \int_G \overline{\phi(hg^{-1})} f_1(h) \overline{f_2(g)}\, dg\, dh.$$

Since these are equal, $T$ is in fact self-adjoint. $\square$

For $g \in G$, $\rho : G \to \mathrm{End}(L^2(G))$ be $(\rho_g f)(h) = f(hg)$ the right-translate of $f$ by $g$.

<u>Propn</u> If $\phi \in C(G)$ and $\lambda \in \mathbb{C}$, then the $\lambda$-eigenspace

$$V(\lambda) = \{ f \in L^2(G) : T_\phi f = \lambda f \}$$

is invariant under $\rho_g$ for all $g \in G$.

<u>Pf</u> Suppose that $T_\phi f = \lambda f$. Then by the definitions we have

$$(T_\phi(\rho_g(f)))(h) = \int_G \phi(xh^{-1}) f(hg)\, dh$$

Change variables $h \to hg^{-1}$ to obtain

$$\int_G \phi(xgh^{-1}) f(h)\, dh = (\rho_g(T_\phi f))(x) = \rho_g(\lambda f)(x) = \lambda \rho_g(f)(x). \quad \square$$

<u>Thm</u> (Peter-Weyl) The matrix coefficients of $G$ are dense in $C(G)$.

<u>Pf</u> Given $f \in C(G)$, one wants to find a matrix coefficient $f'$ such that $|f - f'|_\infty < \varepsilon$ for any $\varepsilon > 0$.

Since $G$ is compact, $f$ is uniformly continuous. This means that $\exists$ $U$ an open nbhd of the identity such that if $g \in U$, then $|\lambda(g) f - f|_\infty < \frac{\varepsilon}{2}$, where $\lambda : G \to \mathrm{End}(C(G))$ is the action induced by left-translation: $(\lambda(g) f)(h) = f(g^{-1}h)$. Let $\phi$ be a real nonnegative function supported in $U$ such that $\int_G \phi(g)\, dg = 1$. Choose $\phi(g) = \phi(g^{-1})$ (by averaging if necessary) such that $T_\phi$ is self-adjoint in addition to compact.

Claim $|T_\phi F - F|_\infty < \frac{\varepsilon}{2}$.

Pf If $h \in G$, we have

$$\left|(\phi * F)(h) - F(h)\right| = \left| \int_G \phi(g) F(g^{-1}h)\, dh - F(h) \overbrace{\int_G \phi(g)\, dg}^{=1} \right|$$

$$= \left| \int_G \left[\phi(g) F(g^{-1}h) - \phi(g) F(h)\right] dg \right|$$

Real, non-negative, & supported on $U$

$$\leq \int_U \phi(g) \left| F(g^{-1}h) - F(h)\right| dg$$

$$\leq \int_U \phi(g) |\lambda(g) F - F|_\infty \, dg$$

$$\leq \int_U \phi(g) \left(\frac{\varepsilon}{2}\right) dg$$

$$= \frac{\varepsilon}{2}$$

Now, we know that $T_\phi$ is a compact operator on $L^2(G)$. If $\lambda$ is an eigenvalue of $T_\phi$, let $V(\lambda)$ be the $\lambda$-eigenspace. By the spectral theorem, the spaces $V(\lambda)$ are finite-dimensional except possibly for $\lambda = 0$, are mutually orthogonal, and span $L^2(G)$ as a Hilbert space. Moreover by the preceding proposition they are also $T_\phi$-invariant. Let $F_\lambda$ be the projection of $f$ onto $V(\lambda)$. Since the $F_\lambda$ are mutually orthogonal, we get $\sum_\lambda |F_\lambda|_2^2 = |F|_2^2 < \infty$.

Now let $F'' = \sum_{|\lambda| > q} F_\lambda$, $F' = T_\phi(F'')$, for some positive real constant $q$.

Now $F''$ and $F'$ are elements of $\bigoplus_{|\lambda|>q} V(\lambda)$, which is finite-dim'l and closed under right-translation. We claim this implies $F''$ and $F'$ are matrix coefficients. To check this, we check the following claim.

<u>Claim</u> (Bump 2.4) Let $f$ be a cts complex-valued function on $G$ compact. The following are equivalent:

① The functions $\lambda_g f$ st $\lambda_g f(x) = f(xg^{-1})$ span a finite-dim'l vector space.

② The functions $\rho_g f$ st $\rho_g f(x) = f(gx)$ span a finite-dim'l vector space.

③ The function $f$ is a matrix coefficient of a finite-dim'l representation of $G$.

<u>PF</u> If $f$ is a matrix coefficient of $(\rho, V)$, so are $\lambda_g f$ and $\rho_g f$, and $\dim(M_\rho) \leq \dim(V)^2$, so ③ $\Rightarrow$ ① and ②. If the functions $\rho_g f$ span a finite-dim'l vector space $V$, then $(\rho, V)$ is a finite-dim'l representation of $G$; moreover if $L: V \longrightarrow \mathbb{C}$, $\phi \longmapsto \phi(1)$ then $f(g) = L(\rho_g f)(g)$, so $f$ is a matrix coefficient of $G$, so ① $\Rightarrow$ ③. ② $\Rightarrow$ ③ is similar. $\square$

So now we know that $F''$ and $F'$ are in fact matrix coefficients.

Since $\sum_{\lambda} |f_\lambda|_2^2 = |f|_2^2 < \infty$, we may choose $q$ st $\sum_{0<q<|\lambda|} |f_\lambda|_2^2$ is less than any small positive number. Therefore we can arrange that

$$\left|\sum_{0<|\lambda|<q} f_\lambda\right|_1 \leq \left|\sum_{0<|\lambda|<q} f_\lambda\right|_2 = \sqrt{\sum_{0<|\lambda|<q} |f_\lambda|_2^2} < \frac{\varepsilon}{2|\phi|_\infty}$$

Now we have

$$T_\phi(f-f'') = T_\phi\Big(f_0 + \sum_{0<|\lambda|<q} f_\lambda\Big) = T_\phi\Big(\sum_{0<|\lambda|<q} f_\lambda\Big)$$

since $f_0 \equiv 0$. But recall that $|T_\phi f|_\infty \leq |\phi|_\infty |f|_1$, so we see that

$$|T_\phi(f-f'')|_\infty = \Big|T_\phi\Big(\sum_{0<|\lambda|<q} f_\lambda\Big)\Big|_\infty \leq |\phi|_\infty \cdot \Big|\sum_{0<|\lambda|<q} f_\lambda\Big|_1 < |\phi|_\infty \cdot \frac{\varepsilon}{2|\phi|_\infty} = \frac{\varepsilon}{2}$$

Now $|f - f'|_\infty = |f - T_\phi f + T_\phi(f - f'')|$ ← Remember $f' = T_\phi(f'')$

$$\leq |f - T_\phi f| + |T_\phi f - T_\phi f''|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \varepsilon$$

And we are done. □

<u>Corollary</u> The matrix coefficients of $G$ are dense in $L^2(G)$.

<u>PF</u> $C(G)$ is dense in $L^2(G)$.

<u>Recall</u> <u>Corollary</u> Let $G$ be a cpt group that has no small subgroups (eg a Lie group). Then $G$ has a faithful finite-dimensional representation.

<u>Equivalent formulation of the Peter-Weyl thm</u> (essentially "representation theory and Fourier analysis are equivalent")

<u>Defn</u> If $H$ is a Hilbert space w/ inner product $\langle , \rangle$, a (cts) representation $\rho: G \to End(H)$ is <u>unitary</u> if $\langle \rho_g v, \rho_g w\rangle = \langle v, w\rangle$ for all $v, w \in H$ and $g \in G$.

<u>Thm</u> (Peter-Weyl) Let $H$ be a Hilbert space and $G$ a compact group. Let $\rho: G \to End(H)$ be a unitary representation. Then $H$ is a direct sum of finite-dim'l representations.

<u>Pf</u> First we check that if $H$ is nonzero it has an irreducible finite-dim'l invt subspace. Pick a nonzero vector $v \in H$, and let $N$ be a neighborhood of the identity in $G$ st if $g \in N$ then $|\rho_g v - v| \leq \frac{|v|}{2}$. We can find some nonnegative continuous function $\phi$ on $G$ supported in $N$ such that $\int_G \phi(g)\,dg = 1$.

<u>Claim</u> $\int_G \phi(g) \rho_g(v)\,dg \neq 0$.

<u>Pf</u> Consider the inner product with $v$. We have that

$$\left\langle \int_G \phi(g) \rho_g(v)\,dg, v\right\rangle = \langle v, v\rangle - \left\langle \int_N \phi(g)(v - \rho_g(v))\,dg, v\right\rangle$$

and we see the second term is

$$\left| \left\langle \int_N \phi(g)(v - \rho_g(v))\, dg, v \right\rangle \right| \leq \int_N |v - \rho_g(v)|\, dg \cdot |v| \leq \frac{|v|^2}{2}$$

So in particular this term cannot cancel w/ $\langle v, v\rangle$ since they have different absolute values. $\square$

Now, by the Peter-Weyl Theorem, we can choose a matrix coefficient $f$ st $|f - \phi|_\infty < \varepsilon$, where $\varepsilon$ is arbitrary. We then have $\int_G (f-\phi)(g)\rho_g(v)\,dg \leq \varepsilon|\rho_g v| = \varepsilon|v|$. So for sufficiently small $\varepsilon$ we can conclude that $\int_G f(g)\rho_g(v)\,dg \neq 0$.

Now since $f$ is a matrix coefficient, so is the function $g \mapsto f(g^{-1})$ (associated to the dual representation). Let $(\sigma, W)$ be a finite-dim'l rep st $\exists$ $w \in W$ and $L: W \to \mathbb{C}$ st $f(g^{-1}) = L(\sigma_g(w))$. Define a map $T: W \to H$ via

$$T(x) = \int_G L(\sigma_{g^{-1}}(x))\, \rho_g(v)\, dg$$

This is easily seen to be an intertwiner by the usual arguments. Moreover it is nonzero since $T(w) = \int_G f(g)\rho_g(v)\,dg \neq 0$. Since $W$ is finite-dim'l, we see that $T(W)$ the image of $W$ is a nonzero finite-dim'l $\rho$-invt subspace of $H$!

So every nonzero unitary representation of $G$ has a nonzero finite-dim'l invt subspace, which we may take to be irreducible.

Now let $(\rho, H)$ be a unitary rep of $G$. Let $\Sigma$ be the set of all sets of orthogonal finite-dim'l irreducible invt subspaces of $H$ ordered by inclusion. Hence if $S \in \Sigma$ and $U, V \in S$, then $U$ and $V$ are finite-dim'l irreducible invt subspaces and either $U = V$ or $U \perp V$. By Zorn's lemma $\Sigma$ has a maximal element $S$. Suppose $S$ does not span $H$ as a Hilbert space. Then let $H'$ be the orthogonal complement of the span of $S$. Then $H'$ contains an invt irreducible subspace, contradicting maximality. Ergo $S$ spans $H$. $\square$