Let $G$ be a top group w/ Haar measure $dh$. Recall that for $F: G \to \mathbb{C}$
we have $|F|_{L^\infty} = \sup_{g \in G} |F(g)|$

$$|F|_p = \left( \int_G |F|^p dh \right)^{1/p}$$

$L^\infty(G)$ - space of measurable functions for which $|F|_{L^\infty} < \infty$.

$L^p(G)$ - space of measurable functions for which $|F|_p < \infty$.

Note that $|F_1| < |F_2| \leq |F|_{L^\infty}$.

First inequality $|F| = \langle |F|, 1 \rangle \leq |F_2| \cdot |1| = |F_2|$

Cauchy - Schwartz

Second inequality $|F_2| = \left( \int_G |F|^2 dh \right)^{1/2} \leq \int \sup \int |F| dh = \sup \int |F(g)| S_0 dh = \sup \int |F(g)| g \in G$

Let $(F * f)(g) = \int_G f(gh^{-1})f(h) dh = \int_G F(h) f(h^{-1}g) dh$.

For $\phi \in C_c(G)$ let $T_\phi$ denote left convolution w/ $\phi$, i.e.

$$(T_\phi f)(g) = \int_G \phi(gh^{-1}) f(h) dh$$
Propn. If \( f \in C(\mathbb{R}) \), then \( T_f \) is a bounded operator on \( L^1(\mathbb{R}) \). If \( \phi \in L^1(\mathbb{R}) \), then \( T_{\phi} F \in L^\infty(\mathbb{R}) \) and \( \| T_{\phi} F \|_{L^\infty} \leq |\phi|_{L^1} \| F \|_{L^1} \).

\[ \| T_{\phi} F \|_{L^\infty} = \sup_{h \in \mathbb{R}} |\int_{\mathbb{R}} \phi(h + \cdot) \cdot F(h) \, dh| \]

\[ \leq \sup_{g \in \mathbb{R}} |\phi(g)| \cdot |g| \| F \|_{L^1} \]

\[ = |\phi|_{L^1} \| F \|_{L^1} \]

So \( |T_{\phi} F|_{L^\infty} \leq |\phi|_{L^1} \| F \|_{L^1} \). For all \( f \in L^\infty(\mathbb{R}), \) \( 1, \mathbb{R} \), \( |T_{\phi} f| \leq |\phi|_{L^1} \| f \|_{L^\infty} \). So \( T \) is a bounded operator in all three metrics.

Propn. If \( f \in C(\mathbb{R}) \), then convolution with \( \phi \) is a bounded operator \( T_f \) on \( L^2(\mathbb{R}) \) and \( \| T_f \| \leq 1 \| f \|_{L^2} \). The operator \( T_f \) is compact, and if \( \phi(\cdot - g) = \phi(\cdot) \), it is furthermore self-adjoint.

PF. Since \( |f|_{L^2} \leq |f|_{L^2} \leq |f|_{L^\infty} \), we have that \( L^\infty(\mathbb{R}) \subseteq L^2(\mathbb{R}) \subseteq L^1(\mathbb{R}) \). By the previous propn, \( \| T_{f} \|_{L^2} \leq \| T_{f} \|_{L^\infty} \leq |\phi|_{L^1} \| f \|_{L^2} \), so the operator norm of \( T_f \) has \( \| T_{f} \| \leq |\phi|_{L^1} \| f \|_{L^2} \).

To show \( T_f \) is compact, by linearity it suffices to check that the image of the unit ball in \( L^2(\mathbb{R}) \) is sequentially compact. Indeed, since the unit ball in \( L^2(\mathbb{R}) \) is contained in the unit ball in \( L^1(\mathbb{R}) \), it suffices to check that \( B = \{ T_{f} F : f \in L^1(\mathbb{R}), \| f \|_{L^1} \leq 1 \} \) is sequentially compact in \( L^2(\mathbb{R}) \). But indeed it suffices to check sequential compactness in \( L^\infty(\mathbb{R}) \), since a sequence that
converges with $1.12$. By the first propn, since $T_g$ is a bounded operator, $B$ is certainly bounded. We will show it is (in the vocabulary from last lecture) equicontinuous. First, since $\Phi$ is continuous and $G$ is compact, $\Phi$ is uniformly continuous. So, given $\epsilon > 0$, we may find a neighborhood $N$ of the identity such that $|\Phi(kg) - \Phi(g)| < \epsilon$ for all $g$ when $k \in N$. Ergo if $f \in L^1(G)$ and $|f|_1 \leq 1$, for any $g \in G$ we have

$$|\Phi(T_gf)(h) - \Phi(T_gf)(g)| = \left| \int_G \Phi(kg^{-1}) - \Phi(gh^{-1}) |f(h)| \, dh \right|$$

$$\leq \int_G |\Phi(kg^{-1}) - \Phi(gh^{-1})| |f(h)| \, dh$$

$$\leq \epsilon |f|_1$$

$$\leq \epsilon$$

Since all $f$'s in $B$ are of the form $\Phi f$, we conclude that $B$ is equicontinuous. Therefore $B$ is sequentially compact by the Arzelà-Ascoli lemma.

For self-adjointness, if $\Phi(g^{-1}) = \overline{\Phi(g)}$, then we have

$$\langle T_g f_1, f_2 \rangle = \int_G \int_G \Phi(h^{-1}) f_1(h) f_2(g) \, dh \, dg$$

and

$$\langle f_1, T_g f_2 \rangle = \int_G \int_G \Phi(h^{-1}) \overline{f_1(h)} f_2(g) \, dh \, dg.$$

Since these are equal, $T$ is in fact self-adjoint. \( \square \)
For \( g \in G \), \( \varphi : G \rightarrow \text{End}(L^2(G)) \) be \((\varphi g)(h) = f(hg)\) the right-translate of \( f \) by \( g \).

**Prop.** If \( \varphi \in \mathcal{C}(G) \) and \( A \in G \), then the \( A \)-eigenspace

\[
V(A) = \sum_{g \in G} \varphi_{g} \in L^2(G) : T_{A} f = Af
\]

is invariant under \( \varphi_{g} \) for all \( g \in G \).

**Pf** Suppose that \( T_{A} f = Af \). Then by the definitions we have

\[
(T_{A} (\varphi_{g}(f)))(h) = \sum_{g \in G} \varphi_{g} \cdot (xg^{-1}) \cdot f(hg) \, dh
\]

Change variables \( h \rightarrow gh^{-1} \) to obtain

\[
\sum_{g \in G} \varphi_{g} \cdot (xg^{-1}) \cdot f(h) \, dh = ((\varphi_{g} T_{A} f)(x)) = \varphi_{g} (Af)(x) = Af_{g}(x).
\]

**Thm. (Peter–Weyl)** The matrix coefficients of \( G \) are dense in \( \mathcal{C}(G) \).

**Pf** Given \( \varphi \in \mathcal{C}(G) \), one wants to find a matrix coefficient \( \varphi' \) such that \( \| \varphi - \varphi' \|_{\infty} < \varepsilon \) for any \( \varepsilon > 0 \).

Since \( G \) is compact, \( \varphi \) is uniformly continuous. This means that \( \exists U \) an open nbhd of the identity such that if \( g \in U \), then

\[
|1(g)\varphi - \varphi|_{\infty} < \frac{\varepsilon}{2},
\]

where \( 1 : G \rightarrow \text{End}(\mathcal{C}(G)) \) is the action induced by left-translation: \((1(g)f)(h) = f(g^{-1}h)\). Let \( \varphi' \) be a real nonnegative function supported in \( U \) such that \( \int_{G} \varphi'(g) \, dg = 1 \).

Choose \( \varphi'(y) = \varphi'(y^{-1}) \) (by averaging if necessary) such that \( T_{\varphi'} \) is self-adjoint in addition to compact.
Claim: \( |T_\varphi f - f|_{\infty} \leq \frac{\varepsilon}{2} \).

Proof: If \( h \in \mathcal{G} \), we have

\[
| (\phi \ast F)(h) - F(h) | = \left| \int_\mathcal{G} \phi(g) F(g \cdot h) \, dg - F(h) \right|
\]

\[
= \left| \int_\mathcal{G} \phi(g) F(g \cdot h) \, dg - \int_\mathcal{G} \phi(g) F(h) \, dg \right|
\]

\[
\leq \int_\mathcal{U} \phi(g) |F(g \cdot h) - F(h)| \, dg
\]

\[
\leq \int_\mathcal{U} \phi(g) |F(h)| \, dg
\]

\[
\leq \frac{\varepsilon}{2}
\]

Now, we know that \( T_\varphi \) is a compact operator on \( L^2(\mathcal{G}) \). If \( \lambda \) is an eigenvalue of \( T_\varphi \), let \( V(\lambda) \) be the \( \lambda \)-eigenspace.

By the spectral theorem, the spaces \( V(\lambda) \) are finite-dimensional except possibly for \( \lambda = 0 \), are mutually orthogonal, and span \( L^2(\mathcal{G}) \) as a Hilbert space. Moreover, by the preceding proposition they are also \( T_\varphi \)-invariant. Let \( f_\lambda \) be the projection of \( f \) onto \( V(\lambda) \). Since the \( f_\lambda \) are mutually orthogonal, we get

\[
\sum_{\lambda \neq 0} |f_\lambda|_2^2 = |f|_2^2 < \infty.
\]

Now let \( f'' = \sum_{\lambda \neq 0} f_\lambda \), \( f' = T_\varphi (f'') \), for some positive real constant \( c \).
Now $F^*$ and $F^\dagger$ are elements of $\mathbb{C} L(V^*)$, which is finite-dimensional and closed under right-translation. We claim this implies $F^*$ and $F^\dagger$ are matrix coefficients. To check this, we check the following claim.

\[ \text{(Claim 2.4)} \]

Let $F$ be a $G$ complex-valued function on $G$ compact. The following are equivalent:

1. The functions $gF$ and $gF(x) = f(xg^{-1})$ span a finite-dimensional vector space.
2. The functions $gF$ and $gF(x) = F(gx)$ span a finite-dimensional vector space.
3. The function $F$ is a matrix coefficient of a finite-dimensional representation of $G$.

\[ \text{PROOF: } \]

If $F$ is a matrix coefficient of $(\rho, V)$, so are $gF$ and $gF$, and $\dim(M_F) \leq \dim(V)^2$, so $\mathbb{C} = \mathbb{C}$, so the functions $gF$ span a finite-dimensional vector space $V$, then $(\rho, V)$ is a finite-dimensional representation of $G$; moreover if $L : V \to \mathbb{C}$, then $F(g) = L(\rho_F)(g)$, so $F$ is a matrix coefficient of $G$, so $\mathbb{C} = \mathbb{C}$, $\mathbb{C} = \mathbb{C}$ is similar.

So now we know that $F^*$ and $F^\dagger$ are in fact matrix coefficients.
Since \( \sum_{x} |f_{x}|^2 = |F|_2^2 < \infty \), we may choose \( \epsilon \) so that \( \sum_{o \in \mathbb{Z} \setminus \{0\}} |f_{o}|^2 \) is less than any small positive number. Therefore we can arrange that

\[
\left| \sum_{o \in \mathbb{Z} \setminus \{0\}} f_{o} \right|_{2} \leq \left| \sum_{o \in \mathbb{Z} \setminus \{0\}} f_{o} \right|_{2} = \sqrt{\sum_{o \in \mathbb{Z} \setminus \{0\}} |f_{o}|^2} < \frac{\epsilon}{21\|f\|_{\infty}}
\]

Now we have

\[
T_{\phi}(f-f') = T_{\phi}(f_{0} + \sum_{o \in \mathbb{Z} \setminus \{0\}} f_{o}) = T_{\phi}\left( \sum_{o \in \mathbb{Z} \setminus \{0\}} f_{o} \right)
\]

Since \( f_{0} = 0 \). But recall that \( 1T_{\phi}F_{\infty} \leq \|f\|_{\infty} \|F\|_{\infty} \), so we see that

\[
|T_{\phi}(f-f')|_{\infty} = |T_{\phi}\left( \sum_{o \in \mathbb{Z} \setminus \{0\}} f_{o} \right)|_{\infty} \leq \|\phi\|_{\infty} \left| \sum_{o \in \mathbb{Z} \setminus \{0\}} f_{o} \right|_{\infty} < \|\phi\|_{\infty} \cdot \frac{\epsilon}{21\|f\|_{\infty}} = \frac{\epsilon}{2}
\]

Now \( |F-f'|_{\infty} = |f - T_{\phi}f + T_{\phi}(f-f')| \leq |f - T_{\phi}f| + |T_{\phi}(f-f')| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \)

And we are done. \( \square \)

**Corollary** The matrix coefficients of \( G \) are dense in \( L^2(\mathbb{Z}) \).

**PF** \( C(\mathbb{Z}) \) is dense in \( L^2(\mathbb{Z}) \).
Recall Corollary let $G$ be a compact group that has no small subgroups (e.g., a Lie group). Then $G$ has a faithful finite-dimensional representation.

Equivalent Formulation of the Peter-Weyl Thm (essentially representation theory and Fourier analysis are equivalent)

Def. If $H$ is a Hilbert space with inner product $< \cdot, \cdot >$, a representation $\rho: G \to \text{End}(H)$ is unitary if $< \rho_g v, \rho_g w > = < v, w >$ for all $v, w \in H$ and $g \in G$.

Thm (Peter-Weyl) Let $H$ be a Hilbert space and $G$ a compact group. Let $\rho: G \to \text{End}(H)$ be a unitary representation. Then $H$ is a direct sum of finite-dimensional representations.

Pf. First, we check that if $H$ is non-zero, it has an irreducible finite-dimensional invariant subspace. Pick a non-zero vector $v \in H$, and let $N$ be a neighborhood of the identity in $G$ so if $g \in N$ then $\| \rho_g v - v \| \leq \frac{1}{2}$. We can find some non-negative continuous function $\phi$ on $G$ supported in $N$ such that $\int_G \phi(g) dg = 1$.

Claim $\int_G \phi(g) \rho_g(v) \rho_g(v)^* dg \neq 0$.

Pf. Consider the inner product with $v$. We have that

$< \int_G \phi(g) \rho_g(v) \rho_g(v)^* dg, v > = < v, v > - \left< \int_N \phi(g) (v - \rho_g(v)) \rho_g(v)^* dg, v > \right>$
and we see the second term is

\[ \left| \langle \sum_{\nu} \phi(g)(\nu - \rho_g(\nu)) \, d\gamma, \nu \rangle \right| \leq \sum_{\nu} |\nu| \cdot |\nu| \cdot \frac{1}{|\nu|} \leq \frac{|\nu|}{2} \]

So in particular this term cannot cancel \( \langle \phi(\nu) \nu \rangle \) since they have different absolute values. \[ \square \]

Now, by the Peter-Weyl Theorem, we can choose a matrix coefficient \( f \) so \( \| f \|_{1 \to 2} \leq \varepsilon \), where \( \varepsilon \) is arbitrary. We then have \[ \sum_{\nu} (F - \eta) \rho_g(\nu) \, d\gamma \leq \varepsilon \| \nu \| = \varepsilon \| \nu \| . \]
So for sufficiently small \( \varepsilon \) we can conclude that \( \sum_{\nu} f(\nu) \rho_g(\nu) \, d\gamma \to 0. \)

Now since \( F \) is a matrix coefficient, so is the function \( g \mapsto F(g^{-1}) \)
(associated to the dual representation). Let \( (\sigma, W) \) be a finite-dimensional rep so \( G \) new and \( L : W \to G \) or \( f(g^{-1}) = L(\sigma_g(w)) \). Define a map \( T : W \to H \) via

\[ T(w) = \sum_{\nu} L(\sigma_{\gamma^{-1}}(\nu)) \rho_g(\nu) \, d\gamma \]

This is easily seen to be an intertwiner by the usual arguments.
Moreover it is nonzero since \( T(w) = \sum_{\nu} f(\nu) \rho_g(\nu) \, d\gamma \neq 0. \) Since \( W \) is finite-dimensional, we see that \( T(w) \) the image of \( W \) is a nonzero finite-dimensional principal subspace of \( H \).

So every nonzero unitary representation of \( G \) has a nonzero finite-dimensional invariant subspace, which we may take to be irreducible.
Now let \((p, H)\) be a unitary rep of \(G\). Let \(\Sigma\) be the set of all sets of orthogonal finite-dimensional irreducible inert subspaces of \(H\) ordered by inclusion. Hence if \(S \in \Sigma\) and \(U, V \in S\), then \(U \cap V\) and \(U \cup V\) are finite-dimensional irreducible inert subspaces and either \(U \cap V = 0\) or \(U \cup V\). By Zorn's lemma \(\Sigma\) has a maximal element \(S\). Suppose \(S\) does not span \(H\) as a Hilbert space. Then let \(H'\) be one orthogonal complement of the span of \(S\). Then \(H'\) contains an inert irreducible subspace, contradicting maximality. Ergo \(S\) spans \(H\). \(\square\)