Lecture 16

References Bump chapters 3 and 4, Kirillov section 4.7

Recall from last time

Thm If F is a matrix coefficient of G and also a class function, then F is a finite linear combination of the characters of irreducible representations.

Goal of today / Wednesday

Thm (Peter-Weyl) The matrix coefficients of G are dense in $C(G)$ (its complex-valued functions on G (and therefore in $L^2(G)$, perhaps more to the point).

Motivating Example $G = S^1 = \mathbb{R}/\mathbb{Z}$

- Haar measure is dx
- Irreps are indexed by $\mathbb{Z}$, $\rho_k(a) = e^{2\pi i k a}$. The corresponding matrix coefficients are the same as the characters $\chi_k(a) = e^{2\pi i k a}$.
- Orthogonality relation is $\int_0^1 e^{2\pi i k x} \overline{e^{2\pi i l x}} = \delta_{kl}$

'Peter-Weyl theorem' in this case just says the functions $e^{2\pi i x}$ are an orthonormal basis for $L^2(S^1, dx)$, which is a main result of Fourier theory. Every L^2-function on S^1 can be written as a series $\sum_{k\in\mathbb{Z}} c_k e^{2\pi i k x}$ which converges in the L^2-metric.

Toward the proof of the Peter-Weyl Thm

Recall A Hilbert space is a real or complex inner product space which is also a complete metric space wrt the distance function induced by the inner product.

Examples

1. $\mathbb{R}^\infty$, $\mathbb{C}^\infty$

2. L^2 space of square-integrable ($\int_X |f|^2 d\mu < \infty$) complex-valued functions on a measure space $(X, \mathcal{M}, \mu)$, w/ inner product $\int_X f(x)\overline{g(x)}\, d\mu$.

Defn If H is a normed vector space, a linear operator $T: h \to h$ is called bounded if $\exists$ a constant C st $|Tx| \leq C|x|$ for all $x \in h$. The smallest such C is the <u>operator norm</u> of T, written $|T|$. T bounded $\Leftrightarrow$ T continuous.

If H is a Hilbert space, a bounded operator T is <u>self-adjoint</u> if $\langle Tf, g\rangle = \langle f, Tg\rangle$.

Exercise If T is bounded and self-adjoint then its eigenvalues are real and eigenspaces corresponding to distinct eigenspaces are orthogonal. Furthermore if $V \subseteq H$ is a subspace st $T(V) \subseteq V$, then $T(V^\perp) \subseteq V^\perp$.

Defn A bounded operator is <u>compact</u> if whenever $\{x_1, x_2, x_3, \ldots\}$ is any bounded sequence in H, the sequence $\{Tx_1, Tx_2, Tx_3, \ldots\}$ has a convergent subsequence.

<u>Thm</u> (Spectral Thm for compact operators) Let T be a compact self-adjoint operator on a Hilbert space H. Let N be the nullspace of T. Then the Hilbert space dimension of $N^\perp$ is at most countable, and $N^\perp$ has an orthonormal basis v_i $(i=1,2,3,\ldots)$ of eigenvectors $Tv_i = \lambda_i v_i$. If $N^\perp$ is not finite-dimensional, then $\lambda_i \to 0$ as $i \to \infty$.

<u>Side Note</u> The notion of an orthonormal basis of a Hilbert space is as follows: an orthonormal basis of a Hilbert space H is a family $\{e_k\}_{k\in B}$ of elements w/ $\langle e_k, e_j\rangle = \delta_{kj}$ whose linear span is dense in H. Then every element in H can we written as the sum of a sequence $\sum c_k e_k$.

<u>Proof of Thm</u>

<u>Claim 1</u> $$|T| = \sup_{x\in H-\{0\}} \frac{|\langle Tx, x\rangle|}{\langle x, x\rangle} \qquad (*)$$

<u>PF</u> For $x \neq 0$, $|\langle Tx, x\rangle| \le |Tx|\cdot|x| \le |T|\cdot|x|^2 = |T|\cdot\langle x,x\rangle$, so $\sup_{x\in H-\{0\}} \frac{|\langle Tx,x\rangle|}{\langle x,x\rangle} \le |T|$.

For the converse, let $\lambda > 0$ be a real constant. Since by self-adjointness $\langle T^2x, x\rangle = \langle Tx, Tx\rangle$, we have

$$\langle Tx, Tx\rangle = \frac{1}{4}\left|\langle T(\lambda x + \lambda^{-1}Tx), \lambda x + \lambda^{-1}Tx\rangle - \langle T(\lambda x - \lambda^{-1}Tx), \lambda x - \lambda^{-1}Tx\rangle\right|$$

↑ writing this out the noncancelling terms work out to $4\langle Tx, Tx\rangle$

$$\le \frac{1}{4}\left[|\langle T(\lambda x + \lambda^{-1}Tx), \lambda x + \lambda^{-1}Tx\rangle| + |\langle T(\lambda x - \lambda^{-1}Tx), \lambda x - \lambda^{-1}Tx\rangle|\right]$$

$$\le \frac{1}{4}\left[\sup_{y\neq 0}\frac{|\langle Ty,y\rangle|}{\langle y,y\rangle}\langle \lambda x + \lambda^{-1}Tx, \lambda x + \lambda^{-1}Tx\rangle + \sup_{y\neq 0}\frac{|\langle Ty,y\rangle|}{\langle y,y\rangle}\langle \lambda x - \lambda^{-1}Tx, \lambda x - \lambda^{-1}Tx\rangle\right]$$

$$= \frac{1}{4}\left(\sup_{y\neq 0}\frac{|\langle Ty,y\rangle|}{\langle y,y\rangle}\right)\left[2\langle \lambda x, \lambda x\rangle + 2\langle \lambda^{-1}Tx, \lambda^{-1}Tx\rangle\right]$$

$$= \frac{1}{2}\left(\sup_{y \neq 0} \frac{\langle Ty, y\rangle}{\langle y, y\rangle}\right)\left[\lambda^2\langle x, x\rangle + \lambda^{-2}\langle Tx, Tx\rangle\right]$$

So if we let $\lambda = \sqrt{\frac{|Tx|}{|x|}}$ we get

$$|Tx|^2 = \langle Tx, Tx\rangle \leq \frac{1}{2}\left(\sup_{x \neq 0} \frac{\langle Ty, y\rangle}{\langle y, y\rangle}\right)\left[\frac{|Tx|}{|x|}\langle x, x\rangle + \frac{|x|}{|Tx|}\langle Tx, Tx\rangle\right]$$

$$= \frac{1}{2}\left(\sup_{x \neq 0} \frac{\langle Ty, y\rangle}{\langle y, y\rangle}\right)\left(2|x||Tx|\right)$$

$$= \left(\sup_{x \neq 0} \frac{\langle Ty, y\rangle}{\langle y, y\rangle}\right)|x||Tx|$$

$$\Rightarrow |Tx| \leq \left(\sup_{y \neq 0} \frac{\langle Ty, y\rangle}{\langle y, y\rangle}\right)|x| \quad \Rightarrow |T| \leq \left(\sup_{x \neq 0} \frac{\langle Tx, x\rangle}{\langle x, x\rangle}\right). \quad \square$$

<u>Claim 2</u> A nonzero compact self-adjoint operator on a Hilbert space H has a nonzero eigenvector.

<u>PF</u> By claim 1, we may choose a sequence $x_1, x_2, x_3, \dots$ of unit vectors st $|\langle Tx_i, x_i\rangle| \to |T|$. Moreover since $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$, $\langle Tx, x\rangle \in \mathbb{R}$. So we can pass to a subsequence w/ the property that $\langle Tx_i, x_i\rangle \to \lambda$ where $\lambda = \pm|T|$. Now since T is compact, we can find a further subsequence st $Tx_i \to v$ for some vector v. We claim that $x_i \to \lambda^{-1}v$, which makes v an eigenvector of T. Observe that

$$|\langle Tx_i, x_i\rangle| \leq |Tx_i||x_i| = |Tx_i| \leq |T||x_i| = |\lambda|$$

Unit vector

So since $\langle Tx_i, x_i\rangle \to \lambda$, $|Tx_i| \to |\lambda|$ as well.

Now we have that

$$|\lambda x_i - Tx_i|^2 = \langle \lambda x_i - Tx_i, \lambda x_i - Tx_i \rangle = \lambda^2 |x_i|^2 + |Tx_i|^2 - 2\lambda \langle Tx_i, x_i \rangle$$

Since $|x_i| = 1$, $|Tx_i| \to |\lambda|$, and $\langle Tx_i, x_i \rangle \to \lambda$, this in total converges to $2\lambda^2 - 2\lambda^2 = 0$. So given that $Tx_i \to v$, $\lambda x_i \to v$ as well, so $x_i \to \lambda^{-1} v$. Hence by continuity, $T(v) = \lambda v$. $\square$

Now we are set up to prove the theorem. Since T is self-adjoint, $N^\perp$ is preserved by T. Let Σ be the set of orthonormal subsets of $N^\perp$ whose elements are eigenvectors of T. Ordering these sets by inclusion, Zorn's lemma implies there is a maximal such set S. Let V be the closure of the linear span of S. We want to show $V = N^\perp$, or equivalently that $V^\perp = N$. It's clear that $N \subseteq V^\perp$. In the other direction, since V is preserved by T, $V^\perp$ is preserved by T, so T induces a compact self-adjoint operator on $V^\perp$. We want to show that in fact $T|_{V^\perp} = 0$, so that $V^\perp \subseteq N$. But otherwise $T|_{V^\perp}$ has a nonzero eigenvector by Claim 2, contradicting maximality of S. So $V^\perp = N$ $\Rightarrow V = N^\perp$.

So $N^\perp$ has an orthonormal basis consisting of eigenvectors v_i. Let λ_i be the corresponding eigenvalues. For any $\varepsilon > 0$ only finitely many $|\lambda_i| > \varepsilon$ since otherwise there is some infinite sequence of v_i with $|Tv_i| > \varepsilon$, contradicting compactness of T. Ergo $N^\perp$ is countable-dimensional and the v_i can be arranged in a sequence st the values $|\lambda_i|$ are decreasing and if the sequence is infinite $|\lambda_i| \to 0$. $\square$

Two further needed propositions

<u>Propn</u> Let X and Y be compact topological spaces w/ Y a metric space w/ distance function d. Let U be the set of cts maps $X \to Y$ st $\forall\, x \in X$ and every $\varepsilon > 0$ $\exists$ a nbhd N of x st $d(f(x), f(x')) < \varepsilon$ for all $x' \in N$ and for all $f \in U$. Then every sequence in U has a uniformly convergent subsequence.

"equicontinuity"

<u>Pf</u> Let $S_0 = \{f_1, f_2, f_3, \dots\}$ be a sequence in U. The goal is to construct a subsequence which is uniformly Cauchy, hence convergent. For each $n > 1$, we will construct $S_n = \{f_{n1}, f_{n2}, f_{n3}, \dots\}$ a subsequence of S_{n-1} with the property that $\sup_{x \in X} d(f_{ni}(x), f_{nj}(x)) \leq \frac{1}{n}$. If we assume we've constructed S_{n-1}, start by choosing an open neighborhood N_x of x st $d(f(y), f(x)) \leq \frac{1}{3n}$ for all $y \in N_x$ and all $f \in X$. Since X is compact, we may cover X by a finite number of such sets, let them be $N_{x_1}, \dots, N_{x_m}$. Since Y is compact, the m-tuples $(f_{n-1,i}(x_1), \dots, f_{n-1,i}(x_m))$ in $Y^{\times m}$ have some accumulation point. Ergo, pick a subsequence $\{f_{ni}\}$ with the property that $d(f_{ni}(x_k), f_{nj}(x_k)) \leq \frac{1}{3n}$ for all i,j and $1 \leq k \leq m$. Then for any y, $\exists\, x_k$ st $y \in N_{x_k}$ and furthermore

$$
\begin{aligned}
d(f_{ni}(y), f_{nj}(y)) &\leq d(f_{ni}(y), f_{ni}(x_k)) + d(f_{ni}(x_k), f_{nj}(x_k)) \\
&\quad + d(f_{nj}(y), f_{nj}(x_k)) \\
&\leq \frac{1}{3n} + \frac{1}{3n} + \frac{1}{3n} \\
&= \frac{1}{n}
\end{aligned}
$$

Now that we have the sequences $\{f_{ni}\}$, we consider the diagonal sequence $\{f_{11}, f_{22}, f_{33}, \ldots\}$. This is uniformly Cauchy so since Y is compact, hence complete, it converges.

We can topologize $C(X)$ by giving it the L^∞ norm $|\cdot|_\infty$, or sup norm.

Propn (Arzela-Ascoli) Let X be a compact space and $U \subseteq C(X)$ is a bounded subset st for each $x \in X$ and $\varepsilon > 0$ there is a nbhd N of x st $|f(x) - f(y)| \le \varepsilon$ $\forall$ $y \in N$ and $f \in U$. Then every sequence in U has a uniformly convergent subsequence.

Pf Since U is bounded, $\exists$ a cpt interval Y in $\mathbb{R}$ st all functions on U take values in Y. Then this follows from the preceding propn.

$\square$