Characters and the set of Class Functions

<u>Propn</u> A matrix coefficient of an irrep of (ρ, V) which is also a class function (i.e., invariant under conjugation) is a constant multiple of χ_ρ.

<u>Remark</u> Note that clearly the subspace M_ρ of matrix coefficients is closed under conjugation; if $\phi(g) = L(\rho_g(v))$, then $\phi(hgh^{-1}) = L(\rho_h \rho_g \rho_{h^{-1}}(v)) = (L \circ \rho_h)(\rho_g(\rho_{h^{-1}}(v)))$

<u>Corollary</u> If ϕ is a matrix coefficient of G and also a class ftn, ϕ is a finite linear combination of the characters of irreducible representations.

<u>PF</u> Let ϕ be a matrix coefficient of (ρ, V), and $V = \sum_{i=1}^{k} n_i V_i$ where the V_i are irreps. Then ϕ breaks down as a sum of matrix coefficients ϕ_i of V_i. Since ϕ is conjugation invt and the spaces M_ρ are conjugation invariant and mutually orthogonal, each ϕ_i is conjugation invt. So each ϕ_i is a multiple of χ_{ρ_i}.

<u>Proof of Propn</u>

<u>Lemma</u> Let (ρ, V) be an irrep of G. Then $M_\rho \cong \mathrm{End}_{\mathbb{C}}(V)$. Moreover the representations of $G \times G$ by $\theta(g_1, g_2)\phi(h) = \phi(g_2^{-1} h g_1)$ on M_ρ and $\Pi(g_1, g_2) T = \rho_{g_2}^{-1} \circ T \circ \rho_{g_1}$ on $\mathrm{End}_{\mathbb{C}} V$ are isomorphic.

<u>PF</u> Let $L \in V^*$, $v \in V$. Then let $F_{L,v} = L(\rho_g(v))$. This is bilinear, hence induces a linear map $\sigma : V \otimes V^* \longrightarrow M_\rho$. This is surjective and by general Schur orthogonality if L_i and v_j run through orthonormal bases then in fact the F_{L_i, v_j} are orthonormal, and in particular linearly independent.

So σ is an isomorphism. Moreover $G(g_1,g_2)F_{L,v}(g) = F_{L,v}(g_2^{-1}gg_1) = L(\rho_{g_2^{-1}}\rho_g\rho_{g_1}(v))$

$= F_{\rho_{g_2}^* L, \rho_{g_1}(v)}(g)$ $\Rightarrow$ σ is an intertwiner and $M_\rho \simeq V^* \otimes V$ as $G\times G$-modules.

Moreover there is a $G\times G$-equivariant isomorpism $V^* \otimes V \to \mathrm{End}_{\mathbb{C}}(V)$. $\square$
$(L, v) \mapsto (u \mapsto L_u(v))$

Pf of propn Suppose ϕ is a matrix coefficient of V irreducible, and ϕ is invariant under conjugation. Then ϕ is a $G\times G$-invt vector in $\mathrm{End}_{\mathbb{C}}(V)$. But by Schur's Lemma there is a unique such vector. Ergo, ϕ is a scalar multiple of χ_ρ.

Compare this w/ previously For a finite group, the characters are an orthonormal basis for all of the class functions.

Some consequences (to be proved after break)

Thm (The Peter-Weyl Thm) Let G be a cpt group. The matrix coefficients of G are dense in $C(G)$ (and correspondingly $L^2(G)$).

Sample Corollary

Defn A topological group G has no small subgroups if it has a nbhd U of the identity st the only subgroup of G contained in U is 1.

Examples

1. Lie groups have no small subgroups
2. $GL(n, \mathbb{Z}_p)$, where here $\mathbb{Z}_p$ is the p-adic integers, has a basis of nbhds at the identity consisting of subgroups.

Propn Let G be a cpt group w/ no small subgroups. Then G has a faithful finite-dim'l representation.

Pf Let U be a nbhd of the identity containing no subgroup other than $\{e\}$. Then there is a finite-dim'l cpx representation and a matrix coefficient ϕ st $\phi(e) = 0$ but $|\phi(g)| > 1$ when $g \notin U$. Any matrix coefficient is constant on $\ker(\rho) \Rightarrow \ker(\rho) \subseteq U \Rightarrow \ker(\rho) = \{e\}$ since U contains no other subgroup.

Thm (Peter and Weyl) Let H be a Hilbert space and G a compact group. Let $\rho: G \to \mathrm{End}(H)$ be a unitary representation. Then H is a direct sum of finite-dim'l irreducible representations.

More on Maximal Tori

Consider a torus $(\mathbb{R}/\mathbb{Z})^r$.

Propn Every irreducible complex representation of $(\mathbb{R}/\mathbb{Z})^r$ is of the form

$$(x_1, \dots, x_r) \mapsto e^{2\pi i (\sum k_j x_j)} \qquad (*)$$

where $(k_1, \dots, k_r) \in \mathbb{Z}^r$.

Pf Certainly any irrep is one-dimensional. By standard Fourier analysis, the maps above span $L^2((\mathbb{R}/\mathbb{Z})^r)$. So the character χ of any irrep (which agrees w/ the representation) is not orthogonal to a map of the form $(*)$ for some $(k_1, \dots, k_r)$. Hence, χ agrees with this map.

The underlying real representations are given by

$$(x_1, \dots, x_r) \mapsto \begin{pmatrix} \cos(2\pi \sum k_i x_i) & \sin(2\pi \sum k_i x_i) \\ -\sin(2\pi \sum k_i x_i) & \cos(2\pi \sum k_i x_i) \end{pmatrix} \qquad (\star)$$

Propn These are exactly the nontrivial real irreps of $(\mathbb{R}/\mathbb{Z})^r$. Their complexifications are exactly $(x_1, \dots, x_r) \mapsto \begin{pmatrix} e^{2\pi i \sum k_i x_i} & 0 \\ 0 & e^{-2\pi i \sum k_i x_i} \end{pmatrix}$.

Quick Detour Real irreducible representations.

- Let V be a real irrep of G.

  $\mathrm{End}_G(V)$ is a division ring $\Rightarrow$ $\mathrm{End}_G(V) = \begin{cases} \mathbb{R} \\ \mathbb{C} \\ \mathbb{H} \end{cases}$

- So defining χ_V in the usual way, $(\chi_V, \chi_V) = \begin{cases} 1 \\ 2 \\ 4 \end{cases}$

- Complexifying gives a complex rep w/ the same (real) character function
  - $(\chi_V, \chi_V) = 1 \rightsquigarrow$ Complexification is irreducible (and self dual)
  - $(\chi_V, \chi_V) = 2 \rightsquigarrow$ Complexification is a sum of two irreps w/ conjugate trace functions
  - $(\chi_V, \chi_V) = 4 \rightsquigarrow$ Complexification is a sum of two copies of an irrep of G w/ real trace

    Exercise Four nonisomorphic irreps doesn't work because $\mathrm{End}_{\mathbb{C}}(V \otimes \mathbb{C}) \simeq \mathrm{End}_{\mathbb{R}}(V) \otimes \mathbb{C} \not\simeq \mathbb{C}^4$ as algebras.

Exercise The number of real irreps of a finite group G is the number of self-dual irreps over $\mathbb{C}$ plus half the number which are not self dual.

Defn A generator of a compact torus T is an element t st the smallest closed subgroup of T containing t is T.

Thm (Kronecker) Let $(t_1, \ldots, t_r) \in \mathbb{R}^r$, and let t be the image of this point in $(\mathbb{R}/\mathbb{Z})^r$. Then t is a generator of $T \iff \{1, t_1, \ldots, t_r\}$ are linearly independent over $\mathbb{Q}$.

PF Let H be the closure of the group $\langle t \rangle$ generated by T. Then T/H is a cpt abelian group, so if $T/H \neq \{e\}$ it has a character $\chi : T/H \to \mathbb{C}^\times$. This is equivalently a character on T which is trivial on H, and therefore of the form $(x_1, \ldots, x_r) \mapsto e^{2\pi \sum k_i x_i}$. As $t \in H$, $\sum k_i t_i \in \mathbb{Z} \Rightarrow 1, t_1, \ldots, t_r$ are linearly dependent over $\mathbb{Q}$. So existence of nontrivial characters of T/H is exactly equivalent to linear dependence of $\{1, \ldots, t_r\}$. $\square$

Corollary The generators of a compact torus T are dense in T.

Let T be a maximal torus in a compact Lie group G.
Let $N(T) = \{g \in G : gTg^{-1} = T\}$ be its normalizer.

Claim $N(T)$ is a closed subgroup.

PF If $t \in T$ is a generator, $N(T)$ is the inverse image of T under the map $G \to G$

$g \mapsto gtg^{-1}$.

<u>Propn</u> Let G be a cpt Lie group and T a maximal torus. Then the connected cpt $N(T)^\circ$ of the identity is T itself. The quotient $N(T)/T$ is a finite group.

<u>Pf</u> Recall that $\mathrm{Aut}(T) \cong GL(r, \mathbb{Z})$ is discrete. There is a homomorphism $N(T) \to \mathrm{Aut}(T)$ in which the action is by conjugation. So if $n \in N(T)^\circ$, in fact n commutes with T. If $T \subsetneq N^\circ(T)$, there is a one-parameter family $t \to n(t)$ in $N^\circ(T)$ not contained in T, and the subgroup generated by T and $\{n(t)\}$ is a closed connected abelian subgroup of G. Hence it is a torus, which contradicts maximality. So $T = N(T)^\circ$. The quotient $N(T)^\circ/T$ is discrete and cpt, hence finite. $\square$

<u>Defn</u> $N(T)/T$ is the <u>Weyl group</u> of G wrt T.

<u>Example</u> $G = U(n)$, $T = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} ; |t_1| = \dots = |t_n| = 1 \right\}$

$N(T)$ = Monomial matrices (a single nonzero entry in each row and column)

$N(T)/T = S_n$

<u>Propn</u> Let G be a Lie group and H a closed abelian subgroup. Then H is a Lie subgroup of G.

<u>Pf</u> Let $U \subseteq \mathfrak{g}$ be a nbhd of 0 on which $\exp$ is smooth, and consider

$$\mathfrak{h} = \{x \in \mathfrak{g} : \exp(tx) \in H \ \forall t \in \mathbb{R}\}$$

<u>Claim</u> For $x \in \mathfrak{h}$ and $y \in U$, if $\exp(y) \in H$ the $[x,y] = 0$.

<u>Pf</u> For any $t > 0$, $\exp(tx)$ and $\exp(y) \in H$ commute, so $\exp(y) = \exp(tx)\exp(y)\exp(-tx) = \exp(Ad(tx)y)$. For small enough t, y and $Ad(tx)y$ are in U, so applying log we have $Ad(tx)y = y \Rightarrow ad(tx)y = y$ $\Rightarrow [x,y] = 0$.

Now we claim $\mathfrak{h}$ is an abelian Lie algebra. Clearly closed under scalar multiplication. Moreover if $x, y \in \mathfrak{h}$, then $[x,y] = 0 \Rightarrow \exp(t(x+y)) = \exp(tx)\exp(ty) \in H$ so $\mathfrak{h}$ closed under addition.

<u>Claim</u> $\exists$ a nbhd $V \ni e$ in G st $V \subseteq \exp(U)$ and $V \cap H = \{\exp(x) : x \in \mathfrak{h} \cap \log(V)\}$.

Once we know this, $V \cap H$ is a smooth locally closed submfd $\Rightarrow$ by translation, $H \subseteq G$ is a submanifold.

For any open $V \subseteq \exp(U)$, we have $V \cap H \supseteq \{\exp(x) : x \in \mathfrak{h} \cap \log(V)\}$. If this is proper $\forall V$, $\exists$ a sequence $\{h_n\} \subset H \cap \exp(U)$ st $h_n \to e$ but $\log(h_n) \notin \mathfrak{h}$. We let $\log(h_n) = x_n$, so that $x_n \to 0$.

Now let $\mathfrak{g} = \mathfrak{h} \oplus V$, where V is a vector subspace (maybe not a Lie algebra). Let $x_n = y_n + z_n$ for $y_n \in \mathfrak{h}$, $z_n \in V$. Now $[x_n, y_n] = 0$, so $\exp(z_n) = \exp(x_n - y_n) = \exp(x_n)\exp(-y_n) \in H$. We can replace x_n by z_n and h_n by $\exp(z_n)$ and still have $h_n \to e$ but $\log(h_n) \notin \mathfrak{h}$.

Choose an inner product on $\mathfrak{g}$ st the unit ball lies in U. Then $\left\{\frac{x_n}{|x_n|}\right\}$ has an accumulation point x_∞ on the unit ball in V.

Claim $x_\infty \in \mathfrak{h}$. (A contradiction since $\mathfrak{h} \cap V = \{0\}$.)

Pf Suffices to check that $e^{tx_\infty} \in H$ for $t<1$. For fixed t, let r_n be the smallest integer greater than $\frac{t}{|x_n|}$. Since $x_n \to 0$, $r_n|x_n| \to t$. So $r_n x_n \to t x_\infty$ and $e^{r_n x_n} = (e^{x_n})^{r_n} \in H$ since $e^{x_n} \in H$. $\underbrace{\text{As } H \text{ closed}}_{\text{had to use somewhere}}$, $e^{tx_\infty} \in H$. So H is a Lie group.

<u>Propn</u> Let T be a maximal torus in the compact connected Lie group G, and let $\mathfrak{t}, \mathfrak{g}$ be the Lie algebras of T and G respectively.

(i) Any vector in $\mathfrak{g}$ fixed by $Ad(T)$ is in $\mathfrak{t}$.

(ii) We have $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is invariant under $Ad(T)$. Under the restriction of Ad to T, $\mathfrak{p}$ decomposes into a direct sum of two-dim'l irreps of T.

<u>PF</u> If $X \in \mathfrak{g}$ is fixed by $Ad(T)$, then $\exp(tX)$ is a one-parameter subgroup commuting w/ T. If $X \notin \mathfrak{t}$, then the closure of the group it generates with T is a torus strictly larger than T, contradicting maximality.

Now, since G is cpt, there is an inner product on $\mathfrak{g}$ which is inv't under the action of G by Ad. The orthogonal complement $\mathfrak{p}$ of $\mathfrak{t}$ is invariant under the action of $Ad(T)$ and contains no vectors fixed by $Ad(T)$. So it decomposes into a sum of nontrivial 2-dim'l irreps of T.

<u>Corollary</u> If G is a compact connected Lie group and T a maximal torus, $\dim(G) - \dim(T)$ is even.

<u>PF</u> The orthogonal complement of $\mathfrak{t}$ in $\mathfrak{g}$ has even dimension. □