Orthogonality of Characters

Let G be a cpt Lie group & V a representation. Choose a basis $\{v_1, \dots, v_k\}$ for V and write $\rho_g = \rho(g)$ in this basis. The entries in the corresponding matrix $\rho_{ij}(g)$ are matrix coefficients of G.

- For G a Lie group, clearly smooth.
- For G merely cpt, cts fcns.

Abstractly A matrix coefficient is a map $\phi(g) = L(\rho_g v)$ for $\rho: G \longrightarrow GL(V)$ a representation & $L: V \longrightarrow \mathbb{C}$ is a linear functional. Then $\rho_{ij}(g) = L_i(\rho(g) \cdot e_j)$ where L_i is an ith coordinate fcn.

The matrix coefficients of G naturally form a ring; the pointwise sum and product of matrix coefficients corresponding to ρ and σ are matrix coefficients of $\rho \oplus \sigma$ and $\rho \otimes \sigma$. The subspace of matrix coefficients of (ρ, V), $(\rho^{\oplus n}, V)$ is (M, P)

Then the character $\chi_\rho(g) = tr(\rho(g)) = \sum \rho_{ii}(g)$.

Properties (as previously)

① $\chi_{\rho \oplus \sigma} = \chi_\rho \oplus \chi_\sigma$

② $\chi_{\rho \otimes \sigma} = \chi_\rho \chi_\sigma$

③ $\chi_\rho(ghg^{-1}) = \chi_\rho(h)$

④ $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$ ← Unitary wrt a G-invt inner product, hence all eigenvalues are unit cpx numbers ⟹ $\sum \lambda_i^{-1} = \sum \bar{\lambda}_i$.

⑤ $\chi_{\rho^*}(g) = \overline{\chi_\rho(g)}$

We have an inner product on $\underbrace{C^\infty(G,\mathbb{C}) \subseteq C(G,\mathbb{C}) \subseteq L^2(G,\mathbb{C})}_{\text{dense}}$

↑ square-integrable fcns wrt Haar measure

given by $(F_1, F_2) = \int_G F_1(g)\overline{F_2(g)}\,dg$.

Note that we're now working w/ Hilbert spaces.

<u>Propn</u> Let (ρ, V) & (σ, W) be irreps of G, and F a linear map $F: V \to W$. Consider the map $\widehat{F}: V \to W$ given by $\widehat{F}(v) = \int_G \sigma_{g^{-1}} \circ F \circ \rho_g(v)\,dg$. (We can compute this integral by choosing a basis and computing each coordinate Fcn to $\mathbb{C}$ separately; this is well-defined.) Then if $V \not\cong W$, $\widehat{F} \equiv 0$, and if $V \cong W$, $\widehat{F} = \left(\frac{tr(F)}{\dim(V)}\right) \cdot Id$.

<u>PF</u> $\widehat{F}(v) = \int_G \sigma_{g^{-1}} \circ F \circ \rho_g\,dg$ has $\sigma_{h^{-1}} \widehat{F} \rho_h(v) = \int_G \sigma_h \sigma_g F \circ \rho_{g^{-1}} \circ \rho_{h^{-1}}(v)\,dg = \widehat{F}(v)$, so $\widehat{F}$ is an intertwiner. Hence if $V \not\cong W$, $\widehat{F} \equiv 0$. But if $V \cong W$, $\widehat{F} = \lambda \cdot Id$ for some $\lambda \neq 0$, and $tr(\widehat{F}) = \lambda \cdot \dim V$. But $tr(\rho_{g^{-1}} \circ F \circ \rho_g) = tr(F)$, so after picking a basis for V, $tr(\widehat{F}) = \int_G tr(F)\,dg = tr(F)$. So $\lambda = \frac{tr(F)}{\dim V}$.

<u>Propn</u> Let V and W be irreps of a compact (Lie) group G. Either every matrix coefficient of V is orthogonal to every matrix coefficient of W, or the representations are isomorphic.

PF First, observe that if $\rho: G \to GL(V)$ is a representation and $\langle , \rangle$ is a G-invariant inner product on V, then any linear functional is of the form $v \mapsto \langle v, v' \rangle$ for some $v' \in V$ => any matrix coefficient is of the form $g \mapsto \langle \rho_g(v), v' \rangle$.

Assume we have matrix coefficients $\varphi_1(g) = \langle \rho_g(v), v' \rangle$ of V and $\varphi_2(g) = \langle \sigma_g(w), w' \rangle$, which are not orthogonal. Then in particular

$$\int_G \langle \rho_g(v), v' \rangle \cdot \overline{\langle \sigma_g(w), w' \rangle}\, dg = \int_G \langle \rho_g(v), v' \rangle \cdot \langle \sigma_{g^{-1}}(w'), w \rangle\, dg \neq 0. \quad (*)$$

Let $T(v) = \int_G \langle \rho_g(v), v' \rangle \cdot \sigma_{g^{-1}}(w')$. This is a map of the form $\hat{F} = \int \sigma_{g^{-1}} F \rho_g$ where $f(v) = \langle v, v' \rangle \cdot w'$. But by (*), $\langle T(v), w \rangle \neq 0$. So $T \neq 0$ => T is an isomorphism => $V \cong W$. So if V and W are not isomorphic, certainly all matrix coefficients are orthogonal.

Thm Let $\rho: G \to GL(V)$ be an irrep of G. Then

$$\int_G \chi_\rho(g)\, dg = \begin{cases} 1 & \text{if } \rho \text{ is trivial} \\ 0 & \text{otherwise} \end{cases}$$

PF The character of the trivial rep is $\chi_\varnothing(g) = 1$. So $\int_G \chi_\rho(g)\, dg$ is exactly the inner product of χ_ρ w/ $\chi_\varnothing$, and matrix coefficients of non-isomorphic reps are orthogonal.

<u>Propn</u> If V is a representation of G and χ its character,

$$\int_G \chi_\rho(g)\,dg = \dim V^G$$

<u>Pf</u> Let $V = \oplus V_i$ be a direct sum of irreps. Let χ_i be the character of V_i. $\int_G \chi_\rho(g)\,dg = \sum \int_G \chi_i(g)\,dg = \#\{V_i : V_i \text{ is trivial}\} = \dim(V^G)$.

<u>Thm</u> Let (ρ, V) and (σ, W) be representations of G w/ characters χ_ρ and χ_σ. Then

$$\int_G \chi_\rho(g)\overline{\chi_\sigma(g)}\,dg = \dim \operatorname{Hom}_G(V_1, V_2)$$

If V and W are irreducible, then $\int_G \chi_\rho(g)\overline{\chi_\sigma(g)}\,dg = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$

(Lemma here)

<u>Pf</u> Consider the representation $\pi : G \to GL(\operatorname{Hom}_{\mathbb{C}}(V, W))$ st $\pi(g)(f) = \sigma_g \circ f \circ \rho_g^{-1}$. By the lemma, $\chi_\pi(g) = \chi_2(g)\overline{\chi_1(g)}$. Ω^G is exactly the space of intertwiners, so

$$\int_G \overline{\chi_\rho(g)}\chi_\sigma(g)\,dg = \dim(\operatorname{Hom}_G(V_1, V_2))$$

Since this is real, $\int_G \chi_\rho(g)\overline{\chi_\sigma(g)}\,dg = \dim(\operatorname{Hom}_G(V_1, V_2))$. $\square$

<u>Lemma</u> Define a representation $\Psi : GL(n,\mathbb{C}) \times GL(m,\mathbb{C}) \longrightarrow GL(\mathrm{Mat}_{n\times m}(\mathbb{C}))$ by

$\Psi(g_1,g_2)(A) = g_2 A g_1^{-1}$. Then we claim $tr\,\Psi(g_1,g_2) = tr(g_1^{-1})\,tr(g_2)$.

<u>PF</u> Both these ftns are cts, so since diagonalizable matrices are dense in $GL(n,\mathbb{C})$, we can assume g_1 and g_2 are diagonalizable - indeed, diagonalized, since conjugation changes neither ftn. If $\alpha_1,\dots,\alpha_n$ and $\beta_1,\dots,\beta_m$ are the diagonal entries respectively, the ijth entry of A is multiplied by $\beta_i\alpha_j^{-1}$ $\Rightarrow$ $tr(\Psi(g_1,g_2)) = tr(g_1^{-1})\,tr(g_2)$.

<u>Propn</u> (General Schur orthogonality) Let (ρ,V) be an irrep of G w/ invariant inner product $\langle,\rangle$. Then

$$\int_G \langle \rho_g(v_1),v_2\rangle \overline{\langle \rho_g(v_3),v_4\rangle}\,dg = \frac{1}{\dim V}\langle v_1,v_3\rangle\overline{\langle v_2,v_4\rangle}$$

<u>PF</u> As previously, for fixed v_2,v_4, $T(v_1) = \int_G \langle \rho_g(v_1),v_2\rangle \rho_g^{-1}(v_4)$ is an intertwiner, hence scalar. So $T(v_1) = cv_1 \Rightarrow \int_G \langle \rho_g(v_1),v_2\rangle\overline{\langle \rho_g(v_3),v_4\rangle}\,dg = \int_G \langle\rho_g(v_1),v_2\rangle\langle \rho_g^{-1}(v_4),v_3\rangle\,dg$

$= \langle T(v_1),v_3\rangle = c\langle v_1,v_3\rangle$. Likewise for fixed v_1,v_3 we see that the integral is equal to $c'\langle v_4,v_2\rangle = \int_G \langle \rho_g(v_1),v_2\rangle\overline{\langle \rho_g(v_3),v_4\rangle}\,dg = d\langle v_1,v_3\rangle\overline{\langle v_2,v_4\rangle}$.

But observe that if $v_1,\dots,v_n$ is an orthonormal basis for V,

$$1 = \int_G |x_g|^2\,dg = \sum_{i,j}\int_G \langle \rho_g(v_i),v_i\rangle\langle \rho_g(v_j),v_j\rangle$$

$$= \sum_{i,j} d\langle v_i,v_j\rangle\overline{\langle v_i,v_j\rangle}$$

$$= nd$$

$\Rightarrow d = \frac{1}{n} = \frac{1}{\dim V}$. $\square$