# More on Representations

## Digression Stabilizers and Representations

Recall If $G$ acts on $M$ smoothly and properly, the orbits $G_p \subseteq M$ are diffeomorphic to $G/G_p$, where $G_p$ is the stabilizer $G_p = \{g \in G : g \cdot p = p\}$

Let's think briefly about stabilizers.

'Clearly $G_p$ is a normal subgroup of $G$, and if $\begin{matrix} j : G \longrightarrow M \\ g \longmapsto g \cdot j \end{matrix}$, $Dj$ has constant rank can be locally written $(x_1, ---, x_n) \to (x_1, ---, x_k, 0, ---, 0)$, so $j^{-1}(p)$ is in local coordinates $\{(0, ---, 0, x_{k+1}, ---, x_n)\}$ a regular submanifold. (This is a slight generalization of the regular value thm.) So $G_p$ is a closed Lie subgroup.

What is the Lie algebra of $G_p$? Given $x \in \mathfrak{g}$, consider the path of diffeomorphisms $\begin{matrix} M \longrightarrow M \\ q \longmapsto \exp(tx) \cdot q \end{matrix}$. This gives a path $\gamma_q(t) = \exp(tx) \cdot q$ at every point, and a vector field $Y_x$ on $M$ st $Y_x(q) = \gamma_q'(0)$.

We get a map $\begin{matrix} \mathfrak{g} \longrightarrow Vect(M) \\ x \longrightarrow Y_x \end{matrix}$. The Lie algebra of $G_p$ is then exactly $x \in \mathfrak{g}$ st $Y_x(p) = 0$.

Remark An ideal of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ st for any $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, $[x,y] \in \mathfrak{h}$.

Propn For $H$ an embedded Lie subgroup of $G$, $H$ normal $\Rightarrow$ $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. and $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and $H, G$ connected $\Rightarrow$ $H$ normal.

PF $H$ normal $\Rightarrow$ For $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, $\exp(x)\exp(y)\exp(-x) \in H$ $\Rightarrow$ $Ad_{\exp(x)} y \in \mathfrak{h}$.
But $(Ad_{\exp(x)})_* y = ad(x) y = [x,y]$, so $[x,y] \in \mathfrak{h}$.
$\Leftarrow$ If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $ad(x)$ preserves $\mathfrak{h}$ $\Rightarrow$ $Ad_{\exp(x)}$ preserves $\mathfrak{h}$. So $\exp(x)\exp(y)\exp(-x) \in H$ for elements of the form $\exp(x) \in \mathfrak{g}$, $\exp(y) \in \mathfrak{h}$. But $G, H$ are connected, so these generate the groups $\Rightarrow$ we are done. □

So eg in $G/G_p \simeq G_p$ for a free proper action of $G$, $T_p(G_p) \simeq \mathfrak{g}/\mathfrak{h}$, w/ notation as in previous page.

Some consequences

Corollary 1 Let $f: G_1 \longrightarrow G_2$ be a morphism of real or cpx Lie groups and $f_*: \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ the morphism of Lie algebras. Then $\operatorname{Ker} f$ is a closed Lie subgroup w/ Lie algebra $\operatorname{Ker} f_*$, and the map $G_1/\operatorname{Ker} f \longrightarrow G_2$ is an immersion. If $\operatorname{Im} f$ is a submfd & thus a closed Lie subgroup, we have a Lie group isomorphism $\operatorname{Im} f \simeq G_1/\operatorname{Ker} f$.

PF Let $G_1$ act on $G_2$ by $g \cdot h = f(g) \cdot h$ for $f \in G_1$, $h \in G_2$. Then the stabilizer of $e \in G_2$ is $\operatorname{ker} f$, so it is a closed Lie subgroup w/ Lie algebra $\operatorname{ker} f_*$, and $G_1/\operatorname{ker} f \hookrightarrow G_2$ is an immersion.

Corollary 2 Let $G$ be a connected Lie group. Then its center $Z(G)$ is a closed Lie subgroup w/ Lie algebra $\mathfrak{z}(\mathfrak{g})$. For $G$ not connected, $Z(G)$ is a closed Lie subgroup w/ Lie algebra contained (possibly properly) in $\mathfrak{z}(\mathfrak{g})$.

PF Given $g \in G$ and $x \in \mathfrak{g}$, it follows from $\exp(\mathrm{Ad}_g(tx)) = g\exp(tx)g^{-1}$ that $g$ commutes w/ all $\exp(tx) \iff \mathrm{Ad}_g(x) = x$. So for a connected Lie group, since elements of the form $\exp(tx)$ generate $G$,

$g \in Z(G) \iff g \in \mathrm{Ker}\,\mathrm{Ad}$, where $\mathrm{Ad}: G \longrightarrow GL(\mathfrak{g})$ is the map induced by the adjoint action. So $Z(G)$ is a closed Lie subgroup w/ Lie algebra $\ker(\mathrm{Ad}_*) = \ker(\mathrm{ad}) = \mathfrak{z}(\mathfrak{g})$.

(If $G$ is not connected, it's not necessarily clear that everything in $G$ commutes w/ the subgroup associated to $\mathfrak{z}(\mathfrak{g})$, e.g. $S^1 \subseteq O(2)$.)

The quotient group $G/Z(G)$ is usually called the adjoint group

$$\mathrm{Ad}\,G = G/Z(G) = \mathrm{Im}(\mathrm{Ad}: G \longrightarrow GL(\mathfrak{g}))$$

$$\mathrm{ad}\,\mathfrak{g} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) = \mathrm{Im}(\mathrm{ad}: \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}))$$

Corollary Let $V$ be a representation $\rho$ of a Lie group $G$, and $v \in V$. The stabilizer $G_v$ is a closed Lie subgroup in $G$ w/ Lie algebra $\{x \in \mathfrak{g} : \rho_x(v) = 0\}$.

<u>Example</u> Let $V$ be a vector space over $\mathbb{k}$ w/ bilinear form $B$, and let $O(V,B) = \{g \in GL(V) : B(g.v, g.w) = B(v,w) \ \forall \ v,w\}$ be the symmetry group. This is a Lie group over $\mathbb{k}$ w/ Lie algebra

$\mathfrak{o}(V,B) = \{x \in \mathfrak{gl}(V) : B(x.v, w) + B(v, x.w) = 0\}$

For if we let $G$ act on the space of bilinear forms by $(gF)(v,w) = F(g^{-1}v, g^{-1}w)$, $O(V,B)$ is the stabilizer of $B$, hence a Lie group.

<u>Examples</u>

- $O(p,q)$, $SO(p,q)$
- $Sp(n, \mathbb{C})$

<u>Example</u> For an arbitrary finite-dim'l associative algebra over $k$,

$Aut(A) = \{g \in GL(A) : (ga)\cdot(gb) = g(a\cdot b) \ \forall \ a,b \in A\}$

is a Lie group w/ Lie algebra

$Der\, A = \{x \in \mathfrak{gl}(A) : (x.a)b + a(x.b) = x.(ab) \ \forall \ a,b \in A\}$

derivations

Indeed if $G = GL(A)$ acts on the space of linear maps $A \otimes A \to A$, $(g.F)(a \otimes b) = gF(g^{-1}a \otimes g^{-1}b)$, $Aut\, A = G_\mu$ where $\mu$ is the multiplication.

# Invariant Bilinear Forms & the Haar measure

Recall that a volume form on an n-dimensional manifold consists of a smooth assignment to every point $p \in M$ of an alternating multilinear map $\omega_p : (T_pM)^{\oplus n} \longrightarrow \mathbb{R}$

$$(v_1, \dots, v_n) \longmapsto \omega_p(v_1, \dots, v_n)$$

Multilinear: Linear in each coordinate

Alternating: $\omega_p(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -\omega_p(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$

What would this look like in coordinates?

$U \subseteq \mathbb{R}^n$ $\quad \overset{V}{T_pU} = \{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ $\quad v_i = \sum a_{ij} \frac{\partial}{\partial x_j}$ $\quad dx_j(v_i) = a_{ij}$

$\underset{V^*}{(T_pU)^*} = \{dx_1, \dots, dx_n\}$

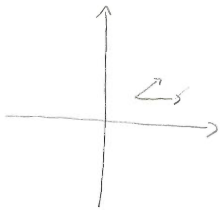
The space of multilinear maps $V^{\otimes k} \to \mathbb{R}$ is generated by tensors $\{dx_{i_1} \otimes \dots \otimes dx_{i_k}\}$, and an arbitrary multilinear map looks like $\sum_{(i_1, \dots, i_k)} a_{(i_1, \dots, i_k)} dx_{i_1} \otimes \dots \otimes dx_{i_k}$. So over $U$ a smooth assignment looks like $\sum_{(i_1, \dots, i_k)} f_{(i_1, \dots, i_k)} dx_{i_1} \otimes \dots \otimes dx_{i_k}$. For the map to be alternating, for $\sigma \in S_k$ we have $f_{(i_{\sigma(1)}, \dots, i_{\sigma(k)})}(p) = \mathrm{sgn}(\sigma) f_{(i_1, \dots, i_k)}(p)$. In particular if $i_k = i_\ell$ for any $k \neq \ell$, $f_{(i_1, \dots, i_k)} \equiv 0$. So if we let $dx_{i_1} \wedge \dots \wedge dx_{i_k} = \sum_{\sigma \in S_k} (-1)^{\mathrm{sgn}\,\sigma} dx_{i_{\sigma(1)}} \wedge \dots \wedge dx_{i_{\sigma(k)}}$, an alternating multilinear map can be written $\sum_{1 \le i_1 < i_2 < \dots < i_k \le n} f_{(i_1,\dots,i_k)}(p)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}$.

The space of alternating multilinear maps $\Lambda^k V^*$ is $\binom{n}{k}$-dimensional.

In particular, in coordinates a volume form is $\omega = f \cdot dx_1 \wedge \dots \wedge dx_n$, where $f$ is some smooth ftn $f: U \to \mathbb{R}$.

Examples

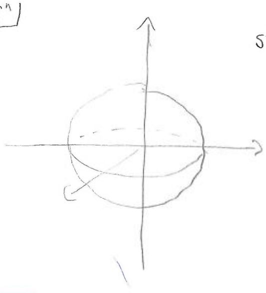
$\mathbb{R}^n$) The standard volume form is $dx_1 \wedge \dots \wedge dx_n$



$\mathbb{R}^2$ $\omega = dx_1 \wedge dx_2$

$\omega((v_1, v_2), (w_1, w_2)) = v_1 w_2 - v_2 w_1$

$S^n$)

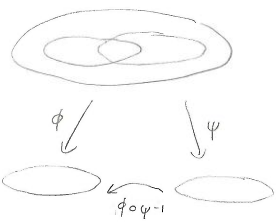


$S^3$ $ds = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy = d\theta \wedge d\phi$

For a hypersurface, we can get a volume form by making the last term a unit normal; $(x, y, z)$ in this case.

(Exercise: Rewrite in your favorite coordinates.)

Change of coordinates

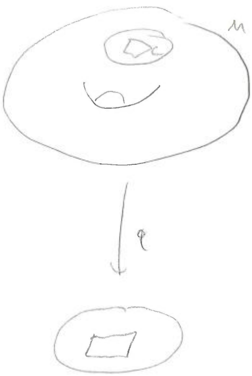


$f\,dx_1 \wedge \dots \wedge dx_n \rightsquigarrow f \circ (\phi \circ \psi^{-1}) \circ \det(\phi \circ \psi^{-1})\, dy_1 \wedge \dots \wedge dy_n$

Note that, in particular, in order for such a form to exist it must be possible to choose coordinate charts for which $\det(\phi \circ \psi^{-1}) > 0$ throughout. This is an orientation.

If $G$ is a Lie group, choosing a basis $\{x_1, \dots, x_n\}$ induces a frame $X_1, \dots, X_n$ st at every $g \in G$ the vectors $X_1(g), \dots, X_n(g)$ are a basis for $T_g G$. So there is a volume form $\omega = dx_1 \wedge \dots \wedge dx_n$.

## Relationship to Integration and measures.




If $C$ cpt and inside a chart on which $\omega = f dx_1 \wedge \dots \wedge dx_n$,

$$\int_C \omega = \int_{\varphi(C)} (\varphi^{-1})^* \omega = \int_{\varphi(C)} f\, dx_1\, dx_2 \cdots dx_n$$

<u>Propn</u> This is well-defined (one checks that the change of variables formula works as expected).

For sets not contained in a single chart (but still cpt) one chooses a partition of unity on $M$:

<u>Defn</u> A partition of unity subordinate to a given atlas is a countable sequence of smooth $\psi_i : M \to [0,1]$ st

1. $\forall\, p \in M$, $\exists$ a nbhd on which $\psi_i = 0$ for all but finitely many $i$.
2. $\sum_{i=1}^{\infty} \psi_i(p) = 1 \ \forall\, p \in M$
3. Each $\psi_i$ has cpt spt contained in some $U_i$ st $\varphi_i : U_i \to V_i$ is a chart in the atlas

Then $\int_C \omega = \sum_i \int_C \psi_i \omega$

Note For $M$ cpt & orientable $\int_M \omega$ is finite.

This defines a Lesbesgue measure $\mu(U) = \int_U \omega = \int_{\bar{U}} \omega$ on the Borel $\sigma$-algebra

(i.e., the $\sigma$-algebra generated from open sets by countable unions, countable intersections, and complements)

For $G$ a Lie group Rescale the left-invt volume form st $\int_G \omega = 1$.

What about right invariance?

- Want to consider $R_g^* \omega$ for all $\omega$.
- There is a map $\phi: G \to \Lambda^n \mathfrak{g}^* \simeq \mathbb{R}^\times \omega_p$. Let $\psi(g)$ be the coefficient
$$g \longmapsto R_g^* \omega_p = \psi(g)\omega_p$$

If $G$ is cpt and connected, we observe that if $|\psi(g)|<1$, then $|\psi(g^n)| = |\psi(g)^n| \to 0$ as $n \to \infty$, which is impossible because $G$ cpt. Likewise if $|\psi(g)| > 0$. So $|\psi(g)| = 1$. $G$ connected $\Rightarrow$ $\omega$ right-invt. $G$ disconnected $\Rightarrow$ $\omega$ right-invt up to a sign.

Inversion? $i: G \to G$. $i^*\omega$ is clearly left-invt since $\omega$ right invt. So suffices to check $\omega$ and $i^*\omega$ are equal up to a sign at $e$. But the derivative of $i$ at $e$ is $-I$, so $i^*\omega = (-1)^n \omega$

Thm Let $G$ be a compact (real Lie) group. Then it has a canonical Borel measure which is left- and right- invt and invt under $g \mapsto g^{-1}$ w/ $\int_G dg = 1$. This is the Haar measure on $G$.

For $F$ cts on $G$, $\int_G F\,dg = \int_G F\cdot\omega$.

Example $G = S^1 \simeq \mathbb{R}/\mathbb{Z}$. The Haar measure is the ordinary measure $dx$

$\omega$ is st
$\omega_e(v) = 1$

Example (we might have the tools to prove at some point)

Let $G = U(n)$ and $F$ be a smooth fcn on $G$ st $F(ghg^{-1}) = F(h)$. Then

$$\int_{U(n)} F(g)\,dg = \frac{1}{n!}\int_T F\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \prod_{i<j} |t_i - t_j|^2\,dt$$

where $T = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, |t_k| = 1 \right\}$ is the subgroup of diagonal matrices

and $dt = \frac{1}{(2\pi)^n} d\theta_1 \cdots d\theta_n$ is the Haar measure.

<u>Thm</u> Any finite-dim'l representation of a cpt Lie group is completely reducible.

<u>PF</u> $\tilde{B}(v,w) = \int_G B(gv, gw)\,dg$. Then $\tilde{B}(v,v) > 0$ since it is a positive ftn

and by right-invariance of the measure, $\tilde{B}(hv, hw) = \int_G B(ghv, ghw)\,dg = \int_G B(gv, gw)\,dg$

$= \tilde{B}(v,w)$.

## Orthogonality of Characters

Let $G$ be a compact real Lie group, and $V$ be a representation. Choose a basis $\{v_1, \dots, v_k\}$ for $V$, and write each $\rho_g = \rho(g)$ in this basis. The entries in the corresponding matrix $\rho_{ij}(g)$ are <u>matrix coefficients</u> of $V$.

As previously, we have an inner product on $C^\infty(G, \mathbb{C})$ given by

$$(F_1, F_2) = \int_G F_1(g)\overline{F_2(g)}\,dg.$$

As previously, the character of a representation $V$ is

$$\chi_\rho(g) = \operatorname{tr}\rho(g) = \sum \rho_{ii}(g)$$

<u>operties</u>

① $\chi_{\rho\oplus\sigma} = \chi_\rho \oplus \chi_\sigma$

② $\chi_{\rho\otimes\sigma} = \chi_\rho \chi_\sigma$

③ $\chi_\rho(ghg^{-1}) = \chi_\rho(h)$

④ $\chi_{\rho^*} = \overline{\chi_\rho}$