Representations of Lie groups & algebras

Defns: A representation of a Lie group G is a vector space V (real or cpx) & a Lie group homomorphism $\rho: G \longrightarrow GL(V)$.

A representation of a Lie algebra $\mathfrak{g}$ is a vector space V w/ a Lie algebra homomorphism $\rho: \mathfrak{g} \to \mathfrak{gl}(V) = GL(V)$ w/ inherited Lie algebra structure.

An intertwining operator (or morphism between representations) is in both cases a linear map $F: V \longrightarrow W$ which commutes w/ the action. We write $\mathrm{Hom}_G(V,W)$ or $\mathrm{Hom}_{\mathfrak{g}}(V,W)$.

Propn ① Every finite-dimensional representation $\rho: G \to GL(V)$ defines a representation $\rho_*: \mathfrak{g} \to \mathfrak{gl}(V)$, and intertwining operators induce intertwining operators.

Also written ρ where no confusion exists.

② If G is connected and simply-ctd, $\rho \mapsto \rho_*$ gives an equivalence of categories of representations of G, and $\mathrm{Hom}_G(V,W) = \mathrm{Hom}_{\mathfrak{g}}(V,W)$

Defn For $\dim(V) < \infty$, the function $\chi_\rho(g) = \mathrm{tr}\, \rho_g$ is the character of π.

Defn As previously, V is irreducible (or simple) if it has no nonzero invt subspaces.
A character is irreducible if it is the character of an irreducible representation.

Remark For $G' = G/Z$, representations of G' are exactly representations of G satisfying $\rho|_Z \equiv id$.

On the homework we saw that real representations induce complex representations. Let's make that precise.

Defn Let $\mathfrak{g}$ be a real Lie algebra. Its complexification is $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}$ w/ obvious commutator. We say $\mathfrak{g}$ is a real form of $\mathfrak{g}_{\mathbb{C}}$.

Example ⓪ If $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$

① If $\mathfrak{g} = \mathfrak{u}(n)$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$
$$\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n) + i\,\mathfrak{u}(n)$$

Related defn Let G be a connected complex Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$, and $K \subset G$ be a closed real Lie subgroup in G st $\mathfrak{k} = \mathrm{Lie}(K)$ is a real form of $\mathfrak{g}$. Then K is a real form of G.

Example $SU(n)$ is a cpt real form of $SL(n, \mathbb{C})$

Note that real forms and complexifications can clearly be quite different!

Why should we care?

Propn Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then a complex representation of $\mathfrak{g}$ induces a representation of $\mathfrak{g}_{\mathbb{C}}$, and $\mathrm{Hom}_{\mathfrak{g}}(V,W) = \mathrm{Hom}_{\mathfrak{g}_{\mathbb{C}}}(V,W)$.

Pf $\rho: \mathfrak{g} \to GL(V)$ induces $\phi: \mathfrak{g}_{\mathbb{C}} \to GL(V)$ w/ $\phi(x+iy) = \rho(x) + i\rho(y)$ exactly as on hw.

Consequence We can sometimes reduce noncompact groups to compact groups, and the representation theory of compact groups is especially nice, so this is very useful

Remark This is really not true for infinite dim'l representations.

Examples of representations we're aware of

① Trivial rep $V = \mathbb{C}$, $\rho_g = \mathrm{id}$ for $g \in G$ ($\rho_x = 0$ for $x \in \mathfrak{g}$)

② Adjoint rep, $V = \mathfrak{g}$, $\rho_g = \mathrm{Ad}_g$, $\rho_x = \mathrm{ad}(x)$

Operations on representations

① W is a G-invariant subspace of $V \iff W$ is a $\mathfrak{g}$-invt subspace of V.

② Given V and W representations of G, we have representations V^*, $V \oplus W$, $V \otimes W$.

What are the corresponding representations of $\mathfrak{g}$?

- Sum $\rho_x(v \oplus w) = \rho_x(v) \oplus \rho_x(w)$

- Tensor Product Note that $\rho_x(v \otimes w) = \rho_x(v) \otimes \rho_x(w)$ is not even a representation of Lie algebras. Taking the derivative of $\rho_g(v \otimes w) = \rho_g(v) \otimes \rho_g(w)$ we see that $\rho_x(v \otimes w) = \rho_x(v) \otimes w + v \otimes \rho_x(w)$.

- Dual For finite-dim'l V, the dual is defined by requiring that $V^* \otimes V \to \mathbb{C}$ be an intertwiner. For groups we want $\langle \rho_g^*(v_1^*), \rho_g(v_2) \rangle = \langle v_2^*, v_1 \rangle$, so $\rho_g^* = (\rho_{g^{-1}})^*$ the adjoint.

Likewise for $\mathfrak{g}$, we want $\langle \rho_x^*(v_2^*), v_1 \rangle + \langle v_2^*, \rho_x(v_1) \rangle = 0$, so $\rho_x^* = -(\rho_x)^*$

Example For $\mathfrak{g}$ a Lie algebra, $\mathfrak{g}^*$ is a representation of $\mathfrak{g}$ given by $\langle ad^*_x(F), y\rangle = -\langle F, ad_x(y)\rangle$ $\quad F \in \mathfrak{g}^*, \; x, y \in \mathfrak{g}$

Defn (as previously) A representation is completely reducible or semisimple if it is isomorphic to a direct sum of irreducibles, $V \simeq \oplus n_i V_i$, V_i irreducibles.

It's of course still not the case that all representations are completely reducible – even complex representations.

Example (Generalizing previous)

$G = \mathbb{R}^+$, $\mathfrak{g} = (\mathbb{R}, [\,,\,] = 0)$

A representation of $\mathfrak{g}$ is a linear map $\begin{array}{l}\mathbb{R} \to End(V) \\ t \mapsto tA\end{array}$ (cpx vector space). The corresponding representation of G is $\begin{array}{l}\mathbb{R} \to End(V) \\ t \mapsto \exp(tA)\end{array}$. Now, any operator in $End(V)$ has an eigenvector v so that $\langle v\rangle$ is fixed by $A \Rightarrow \langle v\rangle$ is a subrepresentation $\Rightarrow$ all complex irreducible reps of $\mathbb{R}$ have dimn 1. So a representation of $\mathbb{R}$ is completely reducible $\Leftrightarrow$ A is diagonalizable, hence not all representations are.

<u>Lemma</u> Let $\rho: G \to GL(V)$ be a representation of G (or $\mathfrak{g}$) and $A: V \to V$ a diagonalizable intertwining operator. Let $V_\lambda \subseteq V$ be the eigenspace for A w/ eigenvalue λ. Then each V_λ is a subrepresentation, and $V = \bigoplus V_\lambda$.

<u>Lemma</u> Let V be a representation of G, $z \in Z(G)$ a central element st ρ_z is diagonalizable. Then $V = \bigoplus V_\lambda$ where V_λ is the eigenspace for $\rho(z)$ w/ eigenvalue λ. Likewise for elements in $\mathfrak{g}$.

<u>Pf</u> If $z \in Z(G)$, $\rho_z \rho_g = \rho_g \rho_z$ $\forall g \in G$, so ρ_z is an intertwiner!

<u>Example</u> $GL(n, \mathbb{C})$ acts on $\mathbb{C}^n \otimes \mathbb{C}^n$. The operator P: which exchanges the factors commutes w/ the action, so $S^2\mathbb{C}^n$, $\Lambda^2\mathbb{C}^n$ subspaces of symmetric- and skew-symmetric tensors, the eigenspaces of P) are $GL(n, \mathbb{C})$-invariant. $\mathbb{C}^n \otimes \mathbb{C}^n = S^2\mathbb{C}^n \oplus \Lambda^2\mathbb{C}^n$. In fact both are irreducible.

As before, we study representations via studying intertwiners.

<u>Schur's Lemma</u> (proof entirely unchanged

1. Let V be an irreducible cpx representation of G. Then $\mathrm{Hom}_G(V, V) = \mathrm{End}(V) = \mathbb{C} \cdot Id$.

2. If V and W are nonisomorphic irreducible cpx representations which are <u>not</u> isomorphic then $\mathrm{Hom}_G(V, W) = 0$.

Likewise for reps of $\mathfrak{g}$.

Examples

(1) If G is commutative, every irreducible complex representation of G is one-dimensional.

Pf In that case, ρ_g is an intertwiner for every G, so in fact $\rho_g = c_g \cdot Id$ for some nonzero constant c_g, so $\rho_g : G \longrightarrow \mathbb{C}^\times$.

(1.5) As observed previously, irreps of $\mathbb{R}$ are maps $\rho : \mathbb{R} \to \mathbb{C}$, $a \mapsto e^{\lambda a}$ for some λ.

Call this representation V_λ.

Representations of $S^1 = \mathbb{R}/\mathbb{Z}$ are representations which satisfy $\rho(a) = 1$ for all $a \in \mathbb{Z}$. Thus we have $V_k = \mathbb{C}$ w/ $\rho : \mathbb{R} \to \mathbb{C}^\times$, $a \mapsto e^{2\pi i k a}$

Identifying $\mathbb{R}/\mathbb{Z}$ w/ $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ this becomes z acts by z^k on V_k.

(2) Group Centers

- $\mathbb{C}^n$ is an irreducible representation of $GL(n, \mathbb{C})$

$\Rightarrow$ every operator commuting w/ the action of $GL(n, \mathbb{C})$ is scalar

$\Rightarrow Z(GL(n,\mathbb{C})) = \{\lambda \cdot Id,\ \lambda \in \mathbb{C}^\times\}$ by Schur's Lemma

Same argument for Lie subgroups $\rightsquigarrow$

$Z(SL(n,\mathbb{C})) = Z(SU(n)) = \{\lambda \cdot Id : \lambda^n = 1\}$, $z(\mathfrak{sl}(n,\mathbb{C})) = z(\mathfrak{su}(n,\mathbb{C})) = 0$

$Z(U(n)) = \{\lambda \cdot Id : |\lambda| = 1\}$, $z(\mathfrak{u}(n)) = \{\lambda \cdot Id : \lambda \in i\mathbb{R}\}$

$$Z(SO(n,\mathbb{C})) = Z(SO(n,\mathbb{R})) = \begin{cases} \{\pm Id\} & n \text{ even},\ n>2 \\ \{Id\} & n \text{ odd} \end{cases}$$

, $z(\mathfrak{so}(n,\mathbb{C})) = z(\mathfrak{so}(n,\mathbb{R})) = 0$

Corollary Let V be a completely reducible representation of a Lie group G $(\sigma, \mathfrak{g})$. Then

① If $V = \oplus V_i$, V_i irreducible and pairwise nonisomorphic, any intertwiner is of the form $\Phi : V \to V$ is $\Phi = \oplus \lambda_i \, Id_{V_i}$.

② If $V = \oplus n_i V_i = \oplus (\mathbb{C}^{n_i} \otimes V_i)$ for V_i irreducible and pairwise nonisomorphic, then an intertwiner $\Phi : V \to V$ is of the form $\Phi = \oplus (A_i \otimes Id_{V_i})$ for $A_i \in End(\mathbb{C}^{n_i})$.

Pf Also same as befor.

Defn A complex representation V of a real Lie group G is unitary if there is a G-invt inner product $(,)$ s.t. $(\rho_g v, \rho_g w) = (v, w)$, or equivalently $\rho_g \in U(V)$ for all $g \in G$.

Likewise a representation V of a real Lie algebra $\mathfrak{g}$ is unitary if there is an inner product which is $\mathfrak{g}$-invt, i.e. $(\rho_x v, w) + (v, \rho_x w) = 0$, or equivalently $\rho_x \in \mathfrak{u}(V)$ $\forall x \in \mathfrak{g}$.

Example We saw finite groups one can always produce a unitary inner product by starting w/ any inner product $B(,)$ and averaging

$$\hat{B}(v, w) = \frac{1}{|G|} \sum_{g \in G} B(gv, gw)$$

Propn (Finite dimensional) unitary representations are always completely reducible.

Pf Exactly the same as previously, given a subrepresentation $W \subseteq V$ it has an orthogonal complement $W^\perp$ which is also a subrepresentation and we may induct on dimension.

New Goal Mimic the averaging operation for groups that aren't finite.

Addition $\rightsquigarrow$ integration

Finite $\rightsquigarrow$ Finite volume.

Propn If G is a locally compact group, there is a unique up to scaling regular Borel measure μ_L which is invt under left translation.

- Borel: The measure is defined on all open sets
- Translation invt $\mu(x) = \mu(gx)$
- Regularity $\mu(x) = \inf\{\mu(U) : U \supseteq x, U \text{ open}\} = \sup\{\mu(K) : K \subseteq x, K \text{ cpt}\}$

This means there is a notion of integration st

$$\int_G f(xg)\, d\mu_L(g) = \int_G f(g)\, d\mu_L(g)$$

In general this is fairly hard. For a smooth Lie group it follows from producing a left-invt volume form.