Switching the derivatives we get $\frac{d}{dt}(\gamma(t)\lambda(u))\cdot\gamma(-t) + \gamma(t)\cdot\frac{d}{dt}(\lambda(u)\gamma(-t))$

$$f \mapsto \frac{d}{du}\frac{d}{dt}\left(F(\gamma(t)\lambda(u)\gamma(-t))\right)\Big|_{t=u=0}$$

$$= \frac{d}{du}\frac{d}{dt}\left(F(\gamma(t)\lambda(u)) - F(\lambda(u)\gamma(t))\right)\Big|_{t=u=0}$$

$$= XY(F) - YX(F)$$

$$= [X,Y](F)$$

The ultimate point is that the linearization of (ABA^{-1}) at identity is $AB-BA$.

Exponentiation and Submanifolds

The fundamental thms of Lie Theory are

Thm1 For any real or complex Lie group, there is a bijection between connected Lie subgroups $H \leq G$ and Lie subalgebras $\mathfrak{h} \leq \mathfrak{g}$, given by $H \to \mathfrak{h} = T_1 H$.

Thm2 If G_1 and G_2 are Lie groups and G is connected & simply ctd, then $\mathrm{Hom}(G_1, G_2) = \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$. (True over $\mathbb{R}$ or $\mathbb{C}$)

Thm3 Any finite-dim'l real or complex Lie algebra is isomorphic to a Lie algebra of a Lie group.

Corollary1 For a real or complex finite dim'l Lie algebra $\mathfrak{g}$, there is a unique (up to isomorphism) connected & simply-ctd Lie group G w/ $\mathrm{Lie}(G) = \mathfrak{g}$. Any other ctd Lie group G' w/ $\mathrm{Lie}(G') = \mathfrak{g}$ is G/Z for Z a discrete central subgroup.

Then let $\varphi: G_1 \xrightarrow{\pi^{-1}} H \hookrightarrow G_1 \times G_2 \xrightarrow{\pi_2} G_2$. This is a morphism of Lie groups w/ $\varphi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ given by the composition $x \mapsto (x, F(x)) \mapsto F(x)$ ◻

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* Lemma Let $F: G_1 \to G_2$ be a morphism of ctd Lie groups st $F_*: T_eG_1 \to T_eG_2$ is an isomorphism, Then f is a covering map, and $\operatorname{Ker} F$ is a discrete central subgroup.

Pf For a Lie group homomorphism, if f_* is an isomorphism at the identity, it is an isomorphism everywhere, so f is a local diffeo everywhere, hence a covering map. The preimage of the identity is necessarily discrete. And it is also certainly normal, and discrete normal subgroups of connected Lie groups are central via considering that for fixed $h \in N$,

$$\begin{aligned} G &\to N \\ g &\mapsto ghg^{-1} \end{aligned}$$

sends $e \mapsto h$ and therefore $g \mapsto h$ for all $g \in G$.

Thm I Relies on the theory of distributions.

Defn Let M be a smooth mfd & TM its tangent bundle. A d-dim'l distribution on M (also called a family) is a d-dimensional subbundle of TM - that is, a choice of a d-dimensional subspace $D_p \subseteq T_pM$ at every x st this choice is made smoothly. (I.e., for every $p \in M$ there is a nbhd U and smooth vector fields $X_1, \dots, X_d$ on U st for $u \in U$ the vectors $X_i(u) \in T_u(M)$ span D_u.

X is subordinate to D if $X(p) \in D_p$ for all $p \in M$

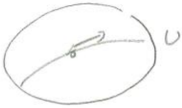
$\mathcal{D}$ is <u>involutory</u> if whenever X, Y are subordinate to $\mathcal{D}$, then $[X, Y]$ is subordinate to $\mathcal{D}$.

<u>Defn</u> An <u>integral manifold</u> of $\mathcal{D}$ is a d-dim'l submanifold $N \subseteq M$ st $\forall\, p \in N$, $T_pN = \mathcal{D}_p$.

<u>Question</u> When is it possible to produce an integral manifold for $\mathcal{D}$ locally/globally?

<u>Observation</u> If we can always choose an integral mfd locally, then $\mathcal{D}$ is involutory.

<u>M</u>]



Let U be an open set small enough that through each pt there is an integral submanifold N.

Let $J \in C^\infty(U)$ be the space of fcns constant on each N_p. Then $X|_{N_p}$ is subordinate to $\mathcal{D} \iff X(j) = 0 \ \forall\, j \in J$. Certainly if $Xj = 0$, $Yj = 0$, $[X, Y]j = 0$ as well. So $[X, Y]$ subordinate to $\mathcal{D}$.

<u>Thm (The Local Frobenius Thm)</u> Let $\mathcal{D}$ be a smooth d-dimensional involutory distribution. Then for each $p \in M$ $\exists$ a nbhd U of x and an integral mfd N of $\mathcal{D}$ through x in U. If N' is another integral mfd through x, then N and N' coincide near x. That is, there is a nbhd V of x st $V \cap N = V \cap N'$.

PF Suffices to check the statement locally. Let $M = \mathbb{R}^n$, $p = 0$.

Let X be a vector field nonvanishing at the origin. To prove the theorem for $d=1$, it suffices to check that $\exists$ coordinates $(y_1, \dots, y_n)$ wrt which $X = \frac{\partial}{\partial y_n}$.

PF of claim Since X nonvanishing at 0, $X(x_i) \neq 0$ at 0 for some coordinate ftn x_i. After possible permutation, $X(x_n) \neq 0$. So if we write

$$X = \sum a_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i},$$

we have $a_n(0) \neq 0$. We switch to coordinates $\vec{y}$ st $(y_1, \dots, y_{n-1}, 0) = (x_1, \dots, x_n, 0)$. For $y_n \neq 0$, consider the integral curve in the direction of X, $t \mapsto (x_1(t), \dots, x_n(t))$ satisfying

$$x_i'(t) = a_i(x_1(t), \dots, x_n(t))$$
$$(x_1(0), \dots, x_n(0)) = (u_1, \dots, u_{n-1}, 0)$$

for small $u_1, \dots, u_{n-1}$. For small enough $u_1, \dots, u_{n-1}$, $a_n(u_1, \dots, u_{n-1}, 0) \neq 0$ so this integral curve points away from (is transverse to) the plane $x_n = 0$. Choose $(y_1, \dots, y_n)$ st

$$y_i(x_1(t), \dots, x_n(t)) = u_i \qquad i = 1, 2, 3, \dots, n-1$$
$$y_n(x_1(t), \dots, x_n(t)) = t$$

Now $\frac{\partial x_i}{\partial y_n} = a_i$. So $\frac{\partial}{\partial y_n} = \sum_i \frac{\partial x_i}{\partial y_n} \frac{\partial}{\partial x_i} = \sum_i a_i \frac{\partial}{\partial x_i} = X$.

For $d=1$ this finishes the result; just take lines in the nth coordinate direction.

For $d>1$, suppose we know that $(d-1)$-dim'l involutory distributions have local submfds. Let $X_1, \dots, X_d$ be smooth vector fields st $X_i(u)$ span D_u near 0. We can assume $X_d = \frac{\partial}{\partial y_n}$. By assumption $[X_d, X_i] = \sum_j g_{ij} X_j \quad (i<d)$

We claim that in fact we can arrange for $g_{id} = 0$ for $i \neq d$.

$$[x_d, x_i] = \sum_{j=1}^{d-1} g_{ij} x_j$$

Indeed, if $x_i = \sum_{k=1}^{n} h_{ik} \frac{\partial}{\partial y_k}$ for $i = 1, \dots, d-1$ we may subtract $h_{in} x_d$ from x_i and still have a spanning set. So we may take $x_i = \sum_{k=1}^{n-1} h_{ik} \frac{\partial}{\partial y_k}$ $i = 1, \dots, d-1$

Then $[x_d, x_i] = \sum_{k=1}^{n-1} \frac{\partial h_{ik}}{\partial y_n} \cdot \frac{\partial}{\partial y_j}$,

But also $[x_d, x_i] = \sum_{j=1}^{d-1} g_{ij} x_i + g_{id} x_d = \sum_{j=1}^{d-1} \sum_{k=1}^{n-1} g_{ij} h_{jk} \frac{\partial}{\partial y_k} + g_{id} \frac{\partial}{\partial y_n}$

This implies $g_{id} = 0$.

Now we want to replace the $x_1, \dots, x_{n-1}$ w/ linear combinations thereof st $[x_i, x_d] = 0$.

<u>Claim</u> Given $(c_1, \dots, c_{d-1})$ real constants, can choose smooth functions f_i st for $y_1, \dots, y_{n-1}$ small, $f_i(y_1, y_2, \dots, y_{n-1}, 0) = c_i$ and $[x_d, \sum_{i=1}^{d-1} f_i x_i] = 0$. This is ode again; consider that

$$\left[x_d, \sum_{i=1}^{d-1} f_i x_i\right] = \sum_{i=1}^{d-1} \frac{\partial f_i}{\partial y_n} x_i + \sum_{i,j=1}^{d-1} f_i g_{ij} x_j$$

To make this zero, we want the f_i to solve the first-order system.

$$\frac{\partial f_j}{\partial y_n} + \sum_{i=1}^{d-1} g_{ij} f_j = 0 \qquad j = 1, \dots, d-1$$

w/ the initial condition above, which is always doable.

Choosing $(c_1, \dots, c_{n-1}) = (1, 0, \dots, 0)$, we get a vector field $\sum f_i x_i$ that agrees w/ x_1 on $y_n = 0$. Replacing x_1 by $\sum f_i x_i$ we can assume $[x_d, x_1] = 0$; iterating we can assume $[x_d, x_i] = 0$ for $i < d$.

Then if $X_i = \sum_{k=1}^{n-1} h_{ik} \frac{\partial}{\partial y_k}$, $\frac{\partial h_{ij}}{\partial y_n} = 0$, so these fcns do not depend

on y_n. Hence the X_i can be interpreted as a $(d-1)$-dim'l involutory distribution in $\mathbb{R}^{n-1}$. By induction $\exists$ an integral mfd for this vector field, $N_0 \subset \mathbb{R}^{n-1}$. We may then take $N = N_0 \times \mathbb{R}$ is an integral mfd for $\mathcal{D}$.

For local uniqueness, iterating the process by which we selected $y_1, \dots, y_n$, we can eventually produce coordinates in which $\mathcal{D}$ is spanned by $\frac{\partial}{\partial y_{n-d+1}}, \dots, \frac{\partial}{\partial y_n}$ and the integral mfd is given by $y_1 = \dots = y_{n-d} = 0$. $\square$

__Defn__ If G is a Lie group, a __local subgroup__ of G consists of an open nbhd U of the identity and a closed subset K of U st $e \in K$, and if $x, y \in K$ and $xy \in U$, then $xy \in K$, and if $x \in K$ st $x^{-1} \in U$, then $x^{-1} \in K$.

__Propn__ Let G be a Lie group and $\mathfrak{h} \subseteq \mathrm{Lie}(G) = \mathfrak{g}$ a Lie subalgebra. There is a local subgroup K of G w/ $T_e K = \mathfrak{h}$, st exp sends a nbhd of 0 in $\mathfrak{h}$ to a nbhd of K.

__Pf__ $\mathfrak{h}$ is involutory, so there is certainly a unique integral mfd in a nbhd U of $\mathfrak{h}$ for $\mathfrak{h}$ at the identity. Let $x, y \in K$ so that $xy \in U$. The image of K under left-translation by x is also an integral mfd of $\mathcal{D}$ through x, so $K = xK$ near x. Since $y \in K$, its left translate xy is also in K.

How does one extend to a global subgroup?

<u>Thm</u> Let D be an involutory distribution on M. Then for every point $p \in M$, $\exists$ a unique connected immersed integral submanifold $N \subseteq M$ of D which contains p and is maximal, i.e. contains any other ctd immersed integral submfd containing p.

<u>Pf</u> Uses theory of flows; see for example Lee's Differential Geometry book.

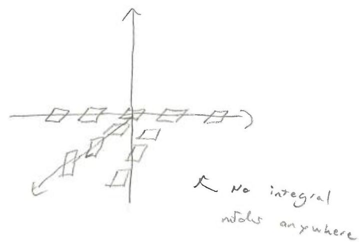
Same arguments w/ translation show that H is a subgroup.

Important
<u>Example</u> $M = \mathbb{R}^3$ $\quad X = \frac{\partial}{\partial x} \quad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \quad V = \text{Span}(X, Y)$, i.e.

$V(x,y,z) = \text{Span}((1,0,0), (0,1,x))$

$$[X,Y]f = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right)f - \left(\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}\right)\frac{\partial}{\partial x} f\Big) = \frac{\partial f}{\partial z}$$

$$\Rightarrow [X,Y] = \frac{\partial}{\partial z}$$



Question: What are the connected abelian Lie groups?

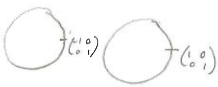
- First question: What are the simply-connected abelian Lie groups?

$$\mathfrak{g} = (\mathbb{R}^n, [\,,\,]) \longleftrightarrow (\mathbb{R}^n, +)$$

- Second question: What are their quotients by discrete central subgroups?

$\mathbb{R}^n/\Lambda \simeq (S^1)^k \times \mathbb{R}^{n-k}$
$\uparrow$
lattice

Remark: Connected is quite important for all this; remember $O(2)$ exists.



Propn: For G any Lie group, and X and Y st $[X,Y]=0$, $\exp(X)\exp(Y) = \exp(X+Y) = \exp(Y)\exp(X)$.

Pf: We saw this previously for matrices, but we can use our computations w/ the adjoint rep to prove it in general. We have $\mathrm{Ad}(\exp(tX))(Y) = \mathrm{ad}(tX)(Y) = [X,Y] = 0$ $\forall\, t \in \mathbb{R}$. So conjugation takes $u \to \exp(uY)$ to itself implying that $\exp(tX)\exp(uY)\exp(-tX) = \exp(uY)$, so $\exp(tX)$ and $\exp(uY)$ commute for all t, u.

Now let $\gamma(u) = \exp(uY)$, $\lambda(t) = \exp(tX)$, so that for $F \in C^\infty(G)$, we have $\frac{d}{du} F(\gamma(u)) = YF(\gamma(u))$ and $\frac{d}{dt} F(\lambda(t)) = XF(\lambda(t))$. Consider the product $q(u,t) = \lambda(t)\gamma(u)$. Since Y is invt under left-translation, $\frac{d}{du} F(q(u,t)) = YF(\lambda(t)\gamma(u))$

Since $\lambda(t)\gamma(u) = \gamma(u)\lambda(t)$, $\frac{d}{dt} F(q(u,t)) = XF(\lambda(t)\gamma(u))$. So

$$\frac{d}{dv}(F(q(v,v))) = \frac{\partial}{\partial t} F(q(t,u))\Big|_{t=u=v} + \frac{\partial}{\partial u} F(q(t,u))\Big|_{t=u=v} = (YF + XF)(q(v,v)).$$

This implies that $\gamma(v)\delta(v)$ is an integral curve for $Y+X$ $\rightsquigarrow$ $\exp(X)\exp(Y) = \exp(X+Y)$.

<u>Corollary</u> For an abelian Lie group the exponential map is a surjective Lie group homomorphism. (Indeed, a covering map $\mathbb{R}^n \longrightarrow \mathbb{R}^k \times (\mathbb{R}/\mathbb{Z})^{n-k}$).

More generally, a <u>torus</u> is any cpt connected abelian Lie group.

<u>Propn</u> Let $T = (\mathbb{R}/\mathbb{Z})^r$. Every homomorphism of T is of the form $t \mapsto Mt \pmod{\mathbb{Z}^r}$ where $M \in GL(r, \mathbb{Z})$. Hence, $\mathrm{Aut}(T) \cong GL(r, \mathbb{Z})$ is discrete.

<u>Pf</u> $\phi : T \longrightarrow T$ induces an invertible linear transformation M of $\mathrm{Lie}(T) = \mathfrak{t}$ commuting w/ the exponential map. Since exp is a homomorphism, ϕ must preserve its kernel $\Lambda = \mathbb{Z}^r$. So $M = GL(r, \mathbb{Z})$.

<u>Remark</u> Any G a cpt ctd Lie group contains (embedded) tori; for example it contains $\{1\}$. An ascending chain of tori $T_1 \subsetneq T_2 \subsetneq T_3$ is bounded above by $\dim G$. Hence G contains (possible multiple) maximal tori.

<u>Example</u> $G = U(n)$ has a maximal torus

$$T = \left\{ \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_n \end{pmatrix} : |t_1| = \dots = |t_n| = 1 \right\}$$

We will come back to understanding maximal tori when we understand something about representations.