This week Relations between Lie groups & Lie algebras

Last Time

$$\exp : M_n(\mathbb{R}) \longrightarrow GL(n,\mathbb{R}) \subseteq M_n(\mathbb{R}) \qquad \text{Likewise } \mathbb{C}$$

Today The general exponential map

Propn Let M be a smooth manifold, X a vector field on M. Then for sufficiently small $\varepsilon > 0$, $\exists$ a path $\gamma : (-\varepsilon, \varepsilon) \to M$ st $\gamma(0) = m$ and $\gamma_*\left(\frac{d}{dt}\right)(t) = X_{\gamma(t)}$ for $t \in (-\varepsilon, \varepsilon)$.

$\gamma(t)$ is an integral curve of the vector field.

Pf In local coordinates let $X = \sum a_i(m) \frac{\partial}{\partial x_i} = \sum a_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}$.

If we pick some smooth path $\gamma(t)$ st $\gamma(0) = m$ w/ coordinates $(x_i(t), \dots, x_n(t))$. Then for $f \in C^\infty(M)$,

$$\gamma_*\left(\frac{d}{dt}\right)(t) f = \frac{d}{dt}(f \circ \gamma) = \sum x_i'(t) \frac{\partial f}{\partial x_i}(x_1(t), \dots, x_n(t))$$

But $$X_{\gamma(t)} f = \sum a_i(x_1(t), \dots, x_n(t)) \frac{\partial f}{\partial x_i}(x_1(t), \dots, x_n(t))$$

$\Rightarrow$ we want a solution to the first order system

$$x_i'(t) = a_i(x_1(t), \dots, x_n(t)) \qquad x_i(0) = 0 \qquad i = 1, \dots, n.$$

It is a standard result from ode that this system has a unique solution. Moreover, if the vector field depends on a smooth parameter, the resulting family of integral curves is smooth.

In general, integral curves only exist on a small nbhd,

Cases when we can always define $\gamma : \mathbb{R} \to M$:

- M is compact
- M is a Lie group

Thm Let G be a Lie group, $\mathfrak{g}$ its Lie algebra. $\exists$ a map $\exp : \mathfrak{g} \to G$ which is a local diffeomorphism near $0 \in \mathfrak{g}$ st $\forall\ X \in \mathfrak{g}$, $t \mapsto \exp(tX)$ is an integral curve for X. Moreover $\exp((t+s)X) = \exp(tX)\exp(sX)$.

PF Let $X \in \mathfrak{g}$. For some $\varepsilon > 0$ $\exists$ $\gamma : (-\varepsilon, \varepsilon) \to G$ for X w/ $\gamma(0) = e$.

First, observe that if $\gamma : (-\varepsilon, \varepsilon) \to G$ is an integral curve on an interval containing 0, for fixed s $t \to \gamma(s+t)$ and $t \to \gamma(s)\gamma(t)$ are both integral curves, in the second case since X is left invt, w/ $0 \to \gamma(s)$ in both cases. By uniqueness $\gamma(s+t) = \gamma(s)\gamma(t)$.

Next, let us show we can extend γ to $(-\frac{3}{2}\varepsilon, \frac{3}{2}\varepsilon)$ and therefore to all of $\mathbb{R}$. We extend by

$$\gamma(t) = \begin{cases} \gamma(\frac{\varepsilon}{2})\gamma(t - \frac{\varepsilon}{2}) & -\frac{\varepsilon}{2} \le t \le \frac{3\varepsilon}{2} \\ \gamma(-\frac{\varepsilon}{2})\gamma(t + \frac{\varepsilon}{2}) & -\frac{3}{2}\varepsilon \le t \le \frac{\varepsilon}{2} \end{cases}$$

By the previous observation this is consistent on regions of overlap.

Let $\exp : \mathfrak{g} \to G$, $X \mapsto \gamma(1)$ where γ is an integral curve for X, $\gamma : \mathbb{R} \to G$, w/ $\gamma(0) = e$. Note that if $s \in \mathbb{R}$, $t \to \gamma(ts)$ is an integral curve for sX, so $\exp(sX) = \gamma(s)$. This map is smooth near the origin by the last observation of the preceding propn. (Indeed, smooth in general, but proving this would take us rather far into the weeds.) Moreover via $T_e\mathfrak{g} \cong \mathfrak{g}$, we have $\exp_* : \mathfrak{g} \to \underset{\overset{\shortparallel}{\mathfrak{g}}}{T_e(G)}$ by identity, so exp is a local diffeomorphism.

Notation Also written e^X.

Remark For the matrix groups, this definition agrees w/ matrix exponentiation; both maps $t \mapsto \exp(t\underset{\substack{\uparrow \\ \text{vector} \\ \text{field or} \\ \text{matrix}}}{X})$ are integral curves for the left-invt vector field defined by X.

Remark Remark that exp is not necessarily surjective (even for connected groups)

eg $\underset{\substack{\shortparallel \\ \left\langle \begin{pmatrix} 0&1\\0&0 \end{pmatrix}, \begin{pmatrix} 0&0\\1&0 \end{pmatrix}, \begin{pmatrix} 1&0\\0&-1 \end{pmatrix} \right\rangle}}{\mathfrak{sl}(2,\mathbb{R})} \longrightarrow SL(2,\mathbb{R}) = \left\{ \begin{pmatrix} a&b\\c&d \end{pmatrix} : ad-bc=1 \right\}$

Im $M = \exp A$ and $J = PAP^{-1}$ is the Jordan form, $\exp(J) = \exp(PAP^{-1}) = PMP^{-1}$

$J = \begin{pmatrix} 0&1\\0&0 \end{pmatrix}$ or $\begin{pmatrix} t&0\\0&-t \end{pmatrix} \rightsquigarrow \exp(J) = \begin{pmatrix} 1&1\\0&1 \end{pmatrix}$ or $\begin{pmatrix} e^t&0\\0&e^{-t} \end{pmatrix}$ is the Jordan form of M.

So we can't hit eg $\begin{pmatrix} -1&1\\0&-1 \end{pmatrix}$.

Remark

- $t \mapsto \exp(tX)$ is a homomorphism for fixed X.
- $\exp(X+Y) \neq \exp(X)\exp(Y)$ in general.

Remark
- $\gamma(t)$ integral curve for X
- $\lambda(t)$ integral curve for Y
- $(\gamma\lambda)'(0) = X+Y$, but does not continue left-invtly in general.

For matrices

$$\exp(A)\exp(B) = \left(I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots\right)\left(I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots\right)$$

$$= I + \underbrace{A + B + AB + \frac{A^2}{2!} + \frac{B^2}{2!} + \cdots}_{X}$$

$$\log(I+X) = \sum_{1}^{\infty} \frac{(-1)^{k+1}X^k}{k}$$

First order terms $A+B$

Second order terms $AB + \frac{A^2}{2} + \frac{B^2}{2} - \frac{1}{2}(A^2 + AB + BA + B^2) = \frac{1}{2}(AB - BA) = \frac{1}{2}[A,B]$

Higher order terms are also expressible in terms of commutators (the Baker-Campbell-Hausdorff Formula)

Likewise in general if we write

$$\exp(X)\exp(Y) = \exp(\mu(X,Y))$$

we have a Taylor series

$$\mu(X,Y) = X + Y + \underset{\substack{\uparrow \\ \text{bilinear,} \\ \text{skew-symmetric}}}{\lambda(X,Y)} + \underset{\substack{\uparrow \\ \text{order} \geq 3 \\ \text{terms}}}{\cdots}$$

and in fact $\lambda(X,Y) = \frac{1}{2}[X,Y]$.

Propn Let G, H be Lie groups w/ Lie algebras $\mathfrak{g}, \mathfrak{h}$. Let $F: G \to H$ be a homomorphism. Then

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{dF} & \mathfrak{h} \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{F} & H \end{array}$$

commutes.

PF F takes an integral curve for a left-invt vector field X on G to an integral curve for F_*X the push-forward.

Interaction w/ the adjoint maps.

Recall $Ad_g : G \to G$
$h \mapsto ghg^{-1}$

This induces a representation $Ad: G \to GL(\mathfrak{g})$
$g \mapsto (Ad_g)_*$

Defn A representation of Lie algebras is a Lie algebra homomorphism $\pi : \mathfrak{g} \to End(V)$

Lie algebra structure inherited from being an algebra. — vector space

$\pi([x,y]) = \pi(x)\pi(y) - \pi(y)\pi(x)$

Let $ad: \mathfrak{g} \to End(\mathfrak{g})$ where $ad(x)(y) = [x,y]$.
$X \mapsto ad(x)$

Propn The representation of Lie algebras induced by Ad is ad.

PF Let $X, Y \in \mathfrak{g} \simeq T_e G$. We have $Ad(g)(Y)(f) = Y(f(ghg^{-1}))$ for $f \in C^\infty(G)$.
Let $\gamma(t) = \exp^{tX}$, so that $\gamma'(0) = X$. Then $ad(X)(Y)$ is the local derivation $f \mapsto \frac{d}{dt}(Ad(\gamma(t))Y)f\big|_{t=0} = \frac{d}{dt}\frac{d}{du} f(\gamma(t)\, \lambda(u)\, \gamma(-t))\big|_{t=u=0}$

$u \mapsto e^{uY}$

Switching the derivatives we get $\frac{d}{dt}(\gamma(t)\lambda(u))\cdot\gamma(-t) + \gamma(t)\cdot\frac{d}{dt}(\lambda(u)\gamma(-t))$

$$f \mapsto \frac{d}{du}\frac{d}{dt}\left(f(\gamma(t)\lambda(u)\gamma(-t))\right)\Big|_{t=u=0}$$

$$= \frac{d}{du}\frac{d}{dt}\left(f(\gamma(t)\lambda(u)) - f(\lambda(u)\gamma(t))\right)\Big|_{t=u=0}$$

$$= XY(f) - YX(f)$$

$$= [X,Y](f)$$

The ultimate point is that the linearization of (ABA^{-1}) at identity is $AB-BA$.

Exponentiation and Submanifolds

The fundamental thms of Lie Theory are

Thm1 For any real or complex Lie group, there is a bijection between connected Lie subgroups $H \leq G$ and Lie subalgebras $\mathfrak{h} \leq \mathfrak{g}$, given by $H \to \mathfrak{h} = T_1 H$.

Thm2 If G_1 and G_2 are Lie groups and G is connected & simply ctd, then $\mathrm{Hom}(G_1, G_2) = \mathrm{Hom}(\mathfrak{g}_1, \mathfrak{g}_2)$. (True over $\mathbb{R}$ or $\mathbb{C}$)

Thm3 Any finite-dim'l real or complex Lie algebra is isomorphic to a Lie algebra of a Lie group.

Corollary1 For a real or complex finite dim'l Lie algebra $\mathfrak{g}$, there is a unique (up to isomorphism) connected & simply-ctd Lie group G w/ $\mathrm{Lie}(G) = \mathfrak{g}$. Any other ctd Lie group G' w/ $\mathrm{Lie}(G') = \mathfrak{g}$ is G/Z for Z a discrete central subgroup.

We will prove some but not all of this.

Working backwards Lemma (*) goes here

Corollary 1 Once we know Thms 2 and 3, we have for any $\mathfrak{g}$ a connected & simply connected Lie group G. Given connected G' w/ $Lie(G') = \mathfrak{g}$ as well, there is a group homomorphism $G \to G'$ which is locally an isomorphism. This is a covering map. Its kernel is a discrete central subgroup.

Thm 3 This relies on Ado's Thm: Any Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{K})$. Requires substantial structure theory.

Simplest case If the center of the Lie algebra is trivial, then

$$\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$
$$x \mapsto ad(x) \neq 0$$

This plus theorem 2 gives the result.

Thm 2 Suppose we have a Lie algebra homomorphism $F: \mathfrak{g}_1 \to \mathfrak{g}_2$. Let $G = G_1 \times G_2$. Its Lie algebra is $\overset{\mathfrak{g}}{\overset{\shortparallel}{\mathfrak{g}_1 \times \mathfrak{g}_2}}$. Let $\mathfrak{h} = \{(x, F(x)) : x \in \mathfrak{g}_1\} \subseteq \mathfrak{g}$

This is a Lie subalgebra since $[(x, F(x)), (y, F(y))] = ([x,y], [F(x), F(y)]) = ([x,y], F[x,y])$

There is a ctd Lie subgroup $H \hookrightarrow G_1 \times G_2$. Projecting to G_1 gives a morphism of Lie groups $\pi: H \to G_1$ and $\pi_*: \mathfrak{h} \to \mathfrak{g}_1$ is an isomorphism

So π is a covering map. But G_1 is simply-ctd and H is ctd, so π is an isomorphism. So $\exists\ \pi^{-1}: G_1 \to H$.

Then let $\varphi : G_1 \xrightarrow{\pi^{-1}} H \hookrightarrow G_1 \times G_2 \xrightarrow{\pi_2} G_2$. This is a morphism of Lie groups w/ $\varphi_* : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$ given by the composition $x \longmapsto (x, F(x)) \mapsto F(x)$ $\square$

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* <u>Lemma</u> Let $f : G_1 \longrightarrow G_2$ be a morphism of ctd Lie groups st $f_* : T_e G_1 \longrightarrow T_e G_2$ is an isomorphism. Then f is a covering map, and $\operatorname{Ker} f$ is a discrete central subgroup.

<u>PF</u> For a Lie group homomorphism, if f_* is an isomorphism at the identity, it is an isomorphism everywhere, so f is a local diffeo everywhere, hence a covering map. The preimage of the identity is necessarily discrete. And it is also certainly normal, and discrete normal subgroups of <u>connected</u> Lie groups are central via considering that for fixed $h \in N$,

$$\begin{aligned} G &\longrightarrow N \\ g &\longmapsto ghg^{-1} \end{aligned}$$

sends $e \mapsto h$ and therefore $g \mapsto h$ for all $g \in G$.

<u>Thm 1</u> Relies on the theory of distributions.

<u>Defn</u> Let M be a smooth mfd & TM its tangent bundle. A d-dim'l <u>distribution</u> on M (also called a <u>family</u>) is a d-dimensional subbundle of TM - that is, a choice of a d-dimensional subspace $D_p \leq T_p M$ at every $\ast$ st this choice is made smoothly. (I.e., for every $p \in M$ there is a nbhd U and smooth vector fields $X_1, \dots, X_d$ on U st for $u \in U$ the vectors $X_i(u) \in T_u(M)$ span D_u.

X is <u>subordinate</u> to D if $X(p) \in D_p$ for all $p \in M$

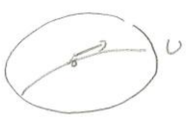
D is <u>involutory</u> if whenever X, Y are subordinate to D, then $[X,Y]$ is subordinate to D.

<u>Defn</u> An <u>integral manifold</u> of D is a d-dim'l submanifold $N \subseteq M$ st $\forall\, p \in N$, $T_pN = D_p$.

<u>Question</u> When is it possible to produce an integral manifold for D locally/globally?

<u>Observation</u> If we can always choose an integral mfd locally, then D is involutory.

<u>M]</u>



Let U be an open set small enough that through each pt there is an integral submanifold N_p. Let $J \in C^\infty(U)$ be the space of ftns constant on each N_p. Then $X|_{N_p}$ is subordinate to D $\Leftrightarrow$ $X(j) = 0$ $\forall\, j \in J$. Certainly if $Xj = 0$, $Yj = 0$, $[X,Y]j = 0$ as well. So $[X,Y]$ subordinate to D.

<u>Thm (The Local Frobenius Thm)</u> Let D be a smooth d-dimensional involutory distribution. Then for each $p \in M$ $\exists$ a nbhd U of x and an integral mfd N of D through x in U. If N' is another integral mfd through x, then N and N' coincide near x. That is, there is a nbhd V of x st $V \cap N = V \cap N'$.

PF Suffices to check the statement locally. Let $M = \mathbb{R}^n$, $p = 0$.

Let X be a vector field nonvanishing at the origin. To prove the theorem for $d = 1$, it suffices to check that $\exists$ coordinates $(y_1, \dots, y_n)$ wrt which $X = \frac{\partial}{\partial y_n}$.

PF of claim Since X nonvanishing at 0, $X(x_i) \neq 0$ at 0 for some coordinate ftn x_i. After possible permutation, $X(x_n) \neq 0$. So if we write

$$X = \sum a_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i},$$

we have $a_n(0) \neq 0$. We switch to coordinates $\vec{y}$ st $(y_1, \dots, y_{n-1}, 0) = (x_1, \dots, x_n, 0)$.
For $y_n \neq 0$, consider the integral curve in the direction of X,
$t \mapsto (x_1(t), \dots, x_n(t))$ satisfying

$$x_i'(t) = a_i(x_1(t), \dots, x_n(t))$$
$$(x_1(0), \dots, x_n(0)) = (u_1, \dots, u_{n-1}, 0)$$

for small $u_1, \dots, u_{n-1}$. For small enough $u_1, \dots, u_{n-1}$, $a_n(u_1, \dots, u_{n-1}, 0) \neq 0$ so this integral curve points away from (is transverse to) the plane $x_n = 0$. Choose $(y_1, \dots, y_n)$ st

$$y_i(x_1(t), \dots, x_n(t)) = u_i \qquad i = 1, 2, 3, \dots, n-1$$
$$y_n(x_1(t), \dots, x_n(t)) = t$$

Now $\frac{\partial x_i}{\partial y_n} = a_i$. So $\frac{\partial}{\partial y_n} = \sum_i \frac{\partial x_i}{\partial y_n} \frac{\partial}{\partial x_i} = \sum_i a_i \frac{\partial}{\partial x_i} = X$.

For $d = 1$ this finishes the result; just take lines in the nth coordinate direction.

For $d > 1$, suppose we know that $(d-1)$-dim'l involutory distributions have local submfds. Let $X_1, \dots, X_d$ be smooth vector fields st $X_i(u)$ span D_u near 0. We can assume $X_d = \frac{\partial}{\partial y_n}$. By assumption $[X_d, X_i] = \sum_j g_{ij} X_j$ $(i < d)$