This week Relations between Lie groups & Lie algebras

Last Time

\[ \text{exp} : M_n(\mathbb{R}) \rightarrow \text{GL}(n, \mathbb{R}) \subseteq M_n(\mathbb{R}) \] likewise &

Today The general exponential map

\[ \text{Prop} \] Let \( M \) be a smooth manifold, \( X \) a vector field on \( M \), then for sufficiently small \( \epsilon > 0 \), there is a path \( \gamma : (-\epsilon, \epsilon) \rightarrow M \) such that \( \gamma(0) = m \) and \( X_m(\frac{d}{dt}) (t) = X_{\gamma(t)} \) for \( t \in (-\epsilon, \epsilon) \).

\( \gamma(t) \) is an integral curve of the vector field.

\[ \text{Def} \] In local coordinates let \( X = \sum a_i(x) \frac{\partial}{\partial x_i} = \sum a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i} \).

If we pick some smooth path \( \gamma(t) \) such that \( \gamma(0) = m \) with coordinates \( (x_1(t), \ldots, x_n(t)) \). Then for \( f \in C^{\infty}(M) \),

\[ \gamma_i \left( \frac{\partial}{\partial t} \right) f = \frac{d}{dt} (f \circ \gamma) = \sum x_i'(t) \frac{\partial f}{\partial x_i} (x_1(t), \ldots, x_n(t)) \]

but \( \gamma_i'(t) = \sum a_i'(x_1(t), \ldots, x_n(t)) \frac{\partial f}{\partial x_i} (x_1(t), \ldots, x_n(t)) \)

\[ \Rightarrow \] we want a solution to the first order system

\[ x_i'(t) = a_i (x_1(t), \ldots, x_n(t)) \quad x_i(0) = 0 \quad i = 1, \ldots, n \]

It is a standard result from ode that this system has a unique solution. Moreover, if the vector field depends on a smooth parameter, the resulting family of integral curves is smooth.
In general, integral curves only exist on a small neighborhood

Cases when we can always define \( \gamma : \mathbb{R} \to M \):

1. \( M \) is compact
2. \( M \) is a Lie group

Then let \( G \) be a Lie group, \( \mathfrak{g} \) its Lie algebra. If a map \( \exp: \mathfrak{g} \to G \)

which is a local diffeomorphism near \( 0 \in \mathbb{R} \) and \( x \in \mathfrak{g} \), \( t \mapsto \exp(tx) \)

is an integral curve for \( X \). Moreover \( \exp((t+s)X) = \exp (tx) \exp(sX) \).

**Proof** Let \( \gamma : \mathbb{R} \to G \) for some \( t > 0 \) \( \gamma : (-t, t) \to G \) for \( X \) \( \gamma(0) = e \).

First, observe that if \( \gamma : (-t, t) \to G \) is an integral curve on

an interval containing \( t \mapsto \gamma(st) \) and \( t \to \gamma(s)\gamma(t) \) are both integral curves in the second case since \( X \) is left invariant, \( \gamma(0) = e \), \( t \to \gamma(t) \) in both cases. By uniqueness \( \gamma(s+e) = \gamma(s)\gamma(e) \).

Next, let us show we can extend \( \gamma \) to \( (-\frac{3}{2}t, \frac{3}{2}t) \) and therefore to all of \( \mathbb{R} \). We extend by \( \tilde{\gamma}(t) = \begin{cases} \gamma(t) & \text{for } -\frac{3}{2}t \leq t \leq \frac{3}{2}t \\ \gamma(-\frac{3}{2}t) & \text{for } -\frac{3}{2}t \leq t \leq \frac{3}{2}t \end{cases} \)

By the previous observation this is consistent on regions of overlap.

Let \( \exp : \mathfrak{g} \to G \) where \( \gamma \) is an integral curve for \( X \), \( \gamma : \mathbb{R} \to G \), \( \gamma(0) = e \).

Note that if \( s \in \mathbb{R} \), \( t \to \gamma(ts) \) is an integral curve for \( sX \),

so \( \exp(sX) = \gamma(s) \). This map is smooth near the origin by the last observation.
of the preceding propn, (Indeed, smooth in general, but proving this would take us rather far into the weeds.) Moreover via 
\( T_0 \cong \mathbb{R} \), we have \( \exp : \mathbb{R} \to T_\mathbb{R}(0) \) by identity, so \( \exp \) is a local diffeomorphism.

**Notation** Also written \( e^x \).

**Remark** For the matrix groups, this definition agrees w/ matrix exponentiation; both maps \( t \mapsto \exp(tx) \) are integral curves for the \[ 1 \text{ vector field or matrix left-invt vector field defined by } X. \]

**Remark** Remark that \( \exp \) is not necessarily surjective (even for connected groups)

\[
\begin{align*}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) & \mapsto \mathbb{R}(a, b) \\
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{align*}
\]

In \( M = \exp A \) and \( J = PAP^{-1} \) is the Jordan form, \( \exp(J) = \exp(PAP^{-1}) = P \exp(A) P^{-1} \).

\( J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \) or \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) \( \mapsto \exp(J) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \) is the Jordan form of \( M \).

So we can't hit \( \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \).
Remark: \( t \mapsto \exp(tX) \) is a homomorphism for fixed \( X \).

\( \exp(x+y) \neq \exp(x) \exp(y) \) in general.

For matrices

\[
\exp(A) \exp(B) = \left( 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots \right) \left( 1 + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \ldots \right)
\]

\[= 1 + (A + B) + AB + \frac{A^2 + B^2}{2!} + \ldots \]

\( \log(I + X) = \sum_{r=1}^{\infty} \frac{(-1)^{k+1}X^k}{r} \)

**First order terms** \( A + B \)

**Second order terms** \[ AB + \frac{A^2 + B^2}{2} - \frac{1}{2} (A^2 + AB + BA + B^2) = \frac{1}{2} (AB - BA) = \frac{1}{2} [A, B] \]

Higher order terms are also expressible in terms of commutators (the Baker-Campbell-Hausdorff Formula).

Likewise in general if we write

\[ \exp(X) \exp(Y) = \exp(m(X, Y)) \]

we have a Taylor series

\[ m(X, Y) = X + Y + \frac{1}{2} \langle X, Y \rangle + \ldots \]

\[ \uparrow \quad \text{bilinear, skew-symmetric terms} \]

\[ \uparrow \quad \text{order } \geq 3 \]

and in fact \( \langle X, Y \rangle = \frac{1}{2} [X, Y] \).
Prop Let \( G, H \) be Lie groups w/ Lie algebras \( g, h \). Let \( F: G \to H \) be a homomorphism. Then \( g \xrightarrow{dF} h \) commutes.
\[
\begin{array}{ccc}
G & \xrightarrow{F} & H \\
\exp & & \exp \\
\end{array}
\]

PF \( F \) takes an integral curve for a left-invariant vector field \( X \) on \( G \) to an integral curve for \( F X \) the push-forward.

Interaction w/ the adjoint maps.

Recall \( \text{Ad}: G \to G \)
\[
h \mapsto ghg^{-1}
\]

This induces a representation \( \text{Ad}: G \to \text{GL}(g) \)
\[
g \mapsto (\text{Ad}_g)^x
\]

Let \( \text{ad}: g \to \text{End}(g) \) where \( \text{ad}(x)(y) = [x,y] \).
\[
X \mapsto \text{ad}(x)
\]

Prop The representation of Lie algebras induced by \( \text{Ad} \) is \( \text{ad} \).

PF Let \( X, Y \in \mathfrak{g} \cong T_e G \). We have \( \text{Ad}(g)(Y)(F) = Y(F(ghg^{-1})) \) for \( F \in C^\infty(G) \).

Let \( \delta(t) = \exp t X \) so that \( \delta'(0) = X \). Then \( \text{ad}(X)(Y) \) is the local derivation
\[
F \mapsto \frac{d}{dt} \big|_{t=0} (\text{Ad}(\delta(t)) Y)
\]

Let \( \frac{d}{dt} \big|_{t=0} (\delta(t) \lambda(u) \delta(t))\big|_{t=0} = \lambda'(u) \]

Refer A representation of

Lie algebras is a

Lie algebra homomorphism

\( \pi: g \to \text{End}(v) \)

\( \pi \) vector space

structure inherited

from being an

algebra.

\[
T[I, Y] = \pi(x) \pi(y) - \pi(y) \pi(x)
\]
Swiching the derivatives we get
\[ \frac{d}{dt}(y(t)X(u)\cdot x(t)) = \frac{d}{dt}(y(t) X(u) \cdot x(t)) \]
\[ = \frac{d}{du} \frac{d}{dt} \left( f \left( x(t) X(u) \cdot x(t) \right) \right) \bigg|_{u=0} \)
\[ = \frac{d}{du} \left( f \left( x(t) X(u) \right) - f(x(t) x(t) \right) \bigg|_{u=0} \)
\[ = \left[ X, y \right](x) \]
\[ = \left[ X, y \right](f) \]

Exponentiation and Submanifolds

The fundamental theorems of Lie theory are

**Thm 1.** For any real or complex Lie group, there is a bijection between connected Lie subgroups \( H \subseteq G \) and Lie subalgebras \( \mathfrak{h} \subseteq \mathfrak{g} \), given by \( H \rightarrow \mathfrak{h} = T_H \).

**Thm 2.** If \( G_1 \) and \( G_2 \) are Lie groups and \( G \) is connected and simply connected, then \( \text{Hom}(G_1, G_2) = \text{Hom}(\mathfrak{g}_1, \mathfrak{g}_2) \). (True over \( \mathbb{R} \) or \( \mathbb{C} \))

**Thm 3.** Any finite-dimensional real or complex Lie algebra is isomorphic to a Lie algebra of a Lie group.

**Corollary.** For a real or complex finite-dimensional Lie algebra \( \mathfrak{g} \), there is a unique (up to isomorphism) connected and simply connected Lie group \( G \) with Lie algebra \( \mathfrak{g} \). Any other connected Lie group \( G' \) with Lie algebra \( \mathfrak{g} \) is \( G' \times \mathbb{Z} \) for a discrete central subgroup \( \mathbb{Z} \).
We will prove some but not all of this.

Working backwards. Lemma (§) goes here.

Corollary 1. Once we know Thm 2 and 3, we have for any \( g \) a connected
\[ \text{simply connected} \] Lie group \( G \). Given connected \( G \) w/ Lie \( (G') = g \) as well,
there is a group homomorphism \( G \rightarrow G' \) which is locally an isomorphism.
This is a covering map. Its kernel is a discrete central subgroup.

Thm 2. This relies on Ado's Thm: Any Lie algebra is isomorphic to a subalgebra
of \( gl(n, \mathbb{K}) \). Requires substantial structure theory.

Simplest case: If the center of the Lie algebra is trivial, then
\[ g \rightarrow gl(g) \]
\[ x \mapsto \text{ad}(x) = 0 \]

This plus Theorem 2 gives the result.

Thm 2. Suppose we have a Lie algebra homomorphism \( \Phi : g_1 \rightarrow g_2 \).
Let \( G = G_1 \times G_2 \). Its Lie algebra is \( g_1 \times g_2 \). Let \( h = g_1 \otimes (\mathbb{K}, f(y)) \); we get \( g_1 \cong g \).
This is a Lie subalgebra since \[ [x, f(y)], [i(x), f(y)] = [i(x, y), f(y)] \]
There is a ctd Lie subgroup \( H \rightarrow G_1 \times G_2 \). Projecting to \( G_1 \) gives
a homorphism of Lie groups \( \pi_1 : H \rightarrow G_1 \), and \( \pi_2 : H \rightarrow G_2 \) is an isomorphism.
So \( \pi_1 \) is a covering map, but \( G_1 \) is simply-ctd and \( H \) is ctd,
so \( \pi_1 \) is an isomorphism. So \( \exists \pi^{-1} : G_1 \rightarrow H \).
Then let $\varphi: G_1 \xrightarrow{\pi_1} H \xrightarrow{\pi_2} G_2$. This is a morphism of Lie groups with $\varphi: y_1 \xrightarrow{\varphi} y_2$ given by the composition $x \mapsto (x, F(x)) \mapsto F(x)$.

*Lemma* Let $\varphi: G_1 \rightarrow G_2$ be a morphism of connected Lie groups with $\varphi: T_0 G_1 \rightarrow T_0 G_2$ an isomorphism. Then $\varphi$ is a covering map, and $\ker\varphi$ is a discrete central subgroup.

**Proof** For a Lie group homomorphism, if $\varphi$ is an isomorphism at the identity, it is an isomorphism everywhere, so $\varphi$ is a local diffeo everywhere, hence a covering map. The preimage of the identity is necessarily discrete.

And it is also certainly normal, and discrete normal subgroups of connected Lie groups are central via considering that for fixed $h \in N$, $g \mapsto ghg^{-1}$ sends $e \mapsto h$ and therefore $g \mapsto h$ for all $g \in G$.

**Thm 1** Relies on the theory of distributions.

**Proof** Let $M$ be a smooth manifold. $TM$ its tangent bundle. A $d$-dimensional distribution on $M$ (also called a family) is a $d$-dimensional subbundle of $TM$—that is, a choice of a $d$-dimensional subspace $P_p \in T_p M$ at every $x \in M$ so this choice is made smoothly. (I.e.) for every $p \in M$ there is a nbhd $U$ and smooth vector fields $X_{i_1}, \ldots, X_{i_d}$ on $U$ so for $u \in U$ the vectors $X_{i_1}(u) \in T_u U$ span $P_u$.

$X$ is subordinate to $P$ if $X(p) \in P_p$ for all $p \in M$. 

$\square$
$D$ is involutory if whenever $X, Y$ are subordinate to $D$, then $[X, Y]$ is subordinate to $D$.

Let an integral manifold of $D$ is a $d$-dim. submanifold $N \subseteq \mathbb{M}$ s.t.

$\forall p \in N, \quad T_p N = D_p$.

**Question** When is it possible to produce an integral manifold for $D$ locally/globally?

**Observation** If we can always choose an integral manifold locally, then $D$ is involutory.

Let $U$ be an open set small enough that through each pt there is an integral submanifold $N$.

Let $f \in C^\infty(U)$ be the space of $f$ is constant on each $N_p$. Then $X_f$ is subordinate to $D$ ($\subset$)

$x(j) = 0 \quad \forall j \in J$. Certainly if $X_j = 0, Y_j = 0, [X,Y]_j = 0$ as well. So $[x,y]$ subordinate to $D$.

**Thm (The Local Frobenius Thm)** Let $D$ be a smooth $d$-dimensional involutory distribution. Then for each $p \in \mathbb{M}$ there is a nbhd $U$ of $x$ and an integral manifold $N$ of $D$ through $x$ in $U$. If $N'$ is another integral manifold through $x$, then $N$ and $N'$ coincide near $x$. That is, there is a nbhd $V$ of $x$ s.t.

$V \cap N = V \cap N'$.
Let $X$ be a vector field nonvanishing at the origin. To prove the theorem for $d=1$, it suffices to check that $d$ coordinates $(y_1, \ldots, y_d)$ wrt which $X = \frac{\partial}{\partial y_d}$.

**Proof of claim** Since $X$, nonvanishing at $0$, $X(0)$ do at $0$ for some coordinate $Frn x_i$. After possible permutation, $X(x_n) \neq 0$. So if we write

$$X = \sum a_i(x_1, \ldots, x_n) \frac{\partial}{\partial x_i},$$

we have $a_n(0) \neq 0$. We switch to coordinates $\tilde{y}$ so $(y_1, \ldots, y_{n-1}, 0) = \tilde{(u_1, \ldots, u_n, 0)}$.

For $\tilde{y} = 0$, consider the integral curve in the direction of $X$,

$$t \rightarrow (x_1(t), \ldots, x_n(t))$$

satisfying

$$x_i'(t) = a_i(x_1(t), \ldots, x_n(t))$$

$$x_n(0) = u_n(0) = 0,$$

for small $u_1, \ldots, u_{n-1}$. For small enough $u_1, \ldots, u_{n-2}$, $a_n(u_1, \ldots, u_{n-1}, 0) \neq 0$ so this integral curve points away from (is transverse to) the plane $x_n = 0$. Choose $(y_1, \ldots, y_n)$ at

$$y_i(x_1(t), \ldots, x_n(t)) = u_i,$$

$$y_n(x_1(t), \ldots, x_n(t)) = t,$$

Now $\frac{\partial x_1}{\partial y_1} = a_1$, so

$$\frac{\partial}{\partial y_1} = \sum_i \frac{\partial x_i}{\partial y_1} \frac{2}{\partial x_i} = \sum_i a_i \frac{\partial}{\partial x_i} = X.$$

For $d=1$ this finishes the result; just take lines in the $n$th coordinate direction.

For $d > 1$, suppose we know that $(d-1)$-dim involute involutive distributions have local subfields. Let $X_1, \ldots, X_d$ be smooth vector fields at $x_i(u)$ span $\kappa_n$ near $0$. We can assume $X_d = \frac{\partial}{\partial x_n}$. By assumption $[X_i, X_j] = \sum g_{ij} X_j$ ($i < d$)