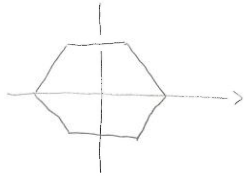
Representation theory of finite groups

- Philosophy: Groups arise naturally as symmetries of objects.
    - S_n is the group of symmetries of $\{1, \dots, n\}$
    - D_{2n} is the symmetries of a regular n-gon



$$D_{12} = \{ r, s : r^6 = 1, s^2 = 1, rs = -sr \}$$

- $O(n)$ is the group of distance-preserving maps of $\mathbb{R}^n$ fixing the origin.
    - Note that this means it is also the group of symmetries of the sphere S^{n-1}
    - $SO(n)$ is its orientation-preserving subgroup

Defn

An action of a group G on a set X is a map $a : G \times X \longrightarrow X$

$$(g, x) \mapsto a(g, x)$$

such that $a(h, a(g,x)) = a(hg, x)$ and $a(e,x) = x$ (ie compatible w/ group law).

One writes $a(g,x) = gx$ since this is unambiguous (ie $h(gx) = (hg)x$)

This is equivalent to specifying a group homomorphism $\alpha : G \longrightarrow \text{Perm}(X)$

$$g \longmapsto \alpha(g).$$

Representation theory starts w/ the group instead of the object and asks "on what spaces does G act?"

Generalized actions is an awful problem.

<u>Defn</u> A linear representation ρ of G on a vector space over a field $\mathbb{k}$ is an action preserving the linear structure

$$\rho(g)(\vec{v}_1 + \vec{v}_2) = \rho(g)\vec{v}_1 + \rho(g)\vec{v}_2 \quad \forall\ \vec{v}_1, \vec{v}_2 \in V$$

$$\rho(g)(k\vec{v}) = k\rho(g)\vec{v} \quad \forall\ k \in \mathbb{k},\ v \in V$$

or equivalently a homomorphism

$$\rho : G \longrightarrow GL(V)$$
$$g \longmapsto \rho(g) = \rho_g$$

In what follows, typically $\mathbb{k} = \mathbb{C}$, $\dim(V) = n < \infty$.

<u>Examples</u>

- G any group, $V = \mathbb{C}$, $\rho : G \longrightarrow GL(\mathbb{C})$, $g \longmapsto Id$ is the <u>trivial rep</u>.

- $G = S_n$, $V = \mathbb{R}$, $\rho : S_n \longrightarrow GL(\mathbb{R})$, $\sigma \longmapsto \mathrm{sgn}(\sigma)$ is the <u>sign rep</u>.

- $G = \mathbb{Z}$, $V = \mathbb{R}^2$, $\rho : \mathbb{Z} \longrightarrow GL(\mathbb{R}^2)$
$$n \longmapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

- $G = S_n$, $V = \mathbb{k}^n$, $\rho_\sigma : \mathbb{k}^n \longrightarrow \mathbb{k}^n$ sends $\rho_\sigma(e_i) = e_{\sigma(i)}$.

permutation representation

- $G = G$; $V = \mathcal{F}(G) = \{\varphi : G \to \mathbb{k}\}$ Functions on G w/ values in $\mathbb{k}$

  w/ $\rho_g(\varphi)(h) = \varphi(hg)$

<u>Exercise</u> Check this is indeed a representation.

- Recall $\mathbb{k}[G]$ the group algebra is the vector space of linear combinations $\sum c_g g$, $c_g \in \mathbb{k}$, w/ the obvious multiplication making it an algebra.

  The <u>regular representation</u> $R : G \to GL(\mathbb{k}[G])$ is $R_h\left(\sum c_g g\right) = \sum c_g hg$

<u>Defn</u> Two representations of G, $\rho : G \to GL(V)$ and $\sigma : G \to GL(W)$ are <u>equivalent</u> or <u>isomorphic</u> if $\exists$ an invertible linear transformation $T : V \to W$ w/ $T \circ \rho_g = \sigma_g \circ T \quad \forall g \in G$

<u>Example</u> For G finite, ρ, and R above are equivalent via

$$T : \mathcal{F}(G) \to \mathbb{k}[G], \quad \varphi \mapsto \sum_{x \in G} \varphi(x) x^{-1}$$

, so that

$$T(\rho_g(\varphi)) = \sum_{x \in G} \varphi(xg) x^{-1} = \sum_{y \in G} \varphi(y) g y^{-1} = R_g(T\varphi)$$

<u>Remark</u> A representation of G is equivalently a module over $\mathbb{k}[G]$.

Ways of producing new representations

Defn If H is a subgroup of G, then $\rho: G \to GL(V)$ induces a restriction $\rho|_H : H \to GL(V)$, written $Res_H \rho$.

Lift If $\varphi: G \to H$ is a homomorphism of groups then $\rho \circ \varphi : G \to GL(V)$ is a representation of G for every $\rho: H \to GL(V)$ a representation of H.
Most commonly: N is a normal subgroup of G and $H = G/N$, w/ φ the projection

Direct sum If we have $\rho: G \to GL(V)$ and $\sigma: G \to GL(W)$, then $\rho \oplus \sigma: G \to GL(V \oplus W)$ maps $(\rho \oplus \sigma)_g (v, w) = (\rho_g(v), \sigma_g(w))$

Tensor Product If we have ρ and σ as above, $\rho \otimes \sigma : G \to GL(V \otimes W)$ via $(\rho \otimes \sigma)_g (v \otimes w) = \rho_g(v) \otimes \sigma_g(w)$.

Dual Let V^* be the dual space of V an $\langle , \rangle : V \otimes V^* \to \mathbb{k}$ be the natural pairing. Given $\rho: G \to GL(V)$ one can define the dual rep $\rho^*: G \to GL(V^*)$ by $\langle \rho_g^* \varrho, v \rangle = \langle \varrho, \rho_g^{-1} v \rangle$

Exercise The regular repn is self-dual for G finite.

Defn An operator $T: V \to W$ is an <u>intertwiner</u> if $T \circ \rho_g = \sigma_g \circ T$ for any $g \in G$. The set of intertwining operators is a vector space $Hom_G(V, W)$. In the case that $\rho = \sigma$, we have $End_G(V,V) = Hom_G(V,V)$ is a $\mathbb{k}$-algebra w/ multiplication given by composition.

Irreducibility

Let $\rho: G \to GL(V)$ be a representation. We say $W \subseteq V$ is G-invariant if $\rho(g)(W) \subseteq W$ for any $g \in G$.

If $W \subseteq V$ is G invariant, we have the subrepresentation $\rho: G \to GL(W)$ and the quotient representation $\rho: G \to GL(V/W)$

Example If $\rho: S_n \to GL(\mathbb{K}^n)$ is the permutation representation, then

$$W = \{x(1,\dots,1) : x \in \mathbb{K}\}$$

$$W' = \{(x_1,\dots,x_n) : x_1 + \dots + x_n = 0\}$$

Thm (Maschke) Let G be a finite group st char k does not divide $|G|$. Let $\rho: G \to GL(V)$ be a representation and $W \subseteq V$ a G-invariant subspace. Then there exists a complementary G-invt subspace, i.e. W' G-invariant st $V = W \oplus W'$.

Pf Let W'' be any subspace st $W \oplus W'' = V$. Let P be the corresponding projection onto W w/ kernel W'', so that $P^2 = P$. Set

$$\bar{P} := \frac{1}{|G|} \sum_{g \in G} \rho_g \circ P \circ \rho_g^{-1}$$

We see that $\rho_g \circ \bar{P} \circ \rho_g^{-1} = \bar{P}$ $\forall g \in G \Rightarrow \rho_g \circ \bar{P} = \bar{P} \circ \rho_g \Rightarrow \bar{P} \in \mathrm{End}_G(V)$. Moreover $\bar{P}|_W = \mathrm{Id}$ and $\mathrm{Im}\,\bar{P} = W$, so $\bar{P}^2 = \bar{P}$. Let $W' = \ker \bar{P}$. Then if $w \in W'$, $\bar{P}(\rho_g w) = \rho_g \bar{P}(w) = \rho_g(0) = 0$, so $\rho_g w \in \ker \bar{P} = W'$. Moreover we claim that $V = W \oplus W'$. Indeed $W \cap W' = 0$ since $\bar{P}|_W = \mathrm{Id}$. And if $v \in V$, we can write $v = \underbrace{\bar{P}v}_{\in W} + \underbrace{(v - \bar{P}v)}_{\in W'}$. So we are done.

Note that we used finiteness and $\frac{1}{|G|}$ making sense ($\operatorname{char} \Bbbk \nmid |G|$).

- $\mathbb{Z} \longrightarrow GL(\mathbb{R}^2)$ has a single invt subspace, $\operatorname{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$

$$n \longmapsto \begin{pmatrix}1 & n\\0 & 1\end{pmatrix}$$

- In the permutation rep, W and W' are complementary $\Leftrightarrow$ $\operatorname{char} \Bbbk \nmid n$.

<u>Defn</u> A nonzero representation is called <u>irreducible</u> if it does not contain any proper nonzero G-invariant subspace.

<u>Exercise</u> The dimension of an irreducible representation of a finite group $|G|$ cannot be larger than G.

<u>Lemma</u> (Schur) Let $\rho: G \longrightarrow GL(V)$ and $\sigma: G \longrightarrow GL(W)$ be two irreducible representations. If $T \in \operatorname{Hom}_G(V, W)$, then $T=0$ or T is an isomorphism.

<u>Pf</u> We have that $\ker T$ and $\operatorname{Im} T$ are G-invt subspaces of V and W respectively. Then by irreducibility of ρ, $\ker T = V$ or $\ker T = 0$. Likewise $\operatorname{Im} T = W$ or $\operatorname{Im} T = 0$. □

<u>Corollary</u> (a) If $\rho: G \longrightarrow GL(W)$ is irreducible, $\operatorname{End}_G(V)$ is a division ring.

(b) If $\operatorname{char} \Bbbk \nmid |G|$, $\operatorname{End}_G(V)$ is a division ring $\Leftrightarrow$ ρ irreducible

(c) If $\Bbbk$ is algebraically closed and ρ is irreducible, $\operatorname{End}_G(V) = \Bbbk$.

(a) Schur's Lemma

(b) If V is reducible, $V = V_1 \oplus V_2$ for some proper invariant subspaces. Let P_1, P_2 be the corresponding projections. Then $P_2 \circ P_1 = 0$, so $\operatorname{End}_G(V)$ has zero divisors.

(c) Consider $T \in \operatorname{End}_G(V)$. Then T has some eigenvalue $\lambda \in \Bbbk$ and $T - \lambda \operatorname{Id} \in \operatorname{End}_G(V)$. $T - \lambda \operatorname{Id}$ is certainly not invertible, so it is zero $\Rightarrow T = \lambda \operatorname{Id}$.

Defn A representation is called <u>completely reducible</u> if it splits into a direct sum of irreducible subrepresentations.

<u>Lemma</u> Let $\rho : G \to GL(V)$ be a representation of G. The following are equivalent

ⓐ ρ is completely reducible.

ⓑ For any G-inv't subspace

Pf For V finite-dimensional, this is easy. For infinite dim'l one has to use Zorn's lemma (the proof is in Gruson - Serganova)

<u>Corollary</u> Let G be a finite group and $\mathbb{k}$ be a field st $\operatorname{char} \mathbb{k} \nmid |G|$. Then every representation of G is completely reducible.

<u>Characters</u>

<u>Defn</u> For a finite-dim'l repn $\rho : G \to GL(V)$ the function $\chi_\rho : G \to \mathbb{k}$ st

$$\chi_\rho(g) = tr(\rho_g)$$

is called the <u>character</u> of the representation.

We easily see that

① $\chi_\rho(1) = \dim \rho$

② If $\rho \cong \sigma$, $\chi_\rho = \chi_\sigma$

③ $\chi_{\rho \oplus \sigma} = \chi_\rho + \chi_\sigma$

④ $\chi_{\rho \otimes \sigma} = \chi_\rho \chi_\sigma$

⑤ $\chi_{\rho^*}(g) = \chi_\rho(g^{-1})$

⑥ $\chi_\rho(ghg^{-1}) = \chi_\rho(h)$

Example For the regular representation $R = \mathbb{k}[G]$, $\chi_R(g) = 0$ for $g \neq 1$ and $\chi_R(1) = |G|$.

Lemma If $\mathbb{k} = \mathbb{C}$ and G finite, for any finite dim'l ρ and $g \in G$ we have $\chi_\rho(g) = \overline{\chi_\rho(g^{-1})}$.

Pf We have $\chi_\rho(g)$ is the sum of the eigenvalues of ρ_g. Since g has finite order, every eigenvalue of ρ_g is a root of 1. The eigenvalues of $\rho_{g^{-1}} = \rho_g^{-1}$ are the complex conjugates.

Orthogonality Let G be finite, $\mathbb{k} = \mathbb{C}$.

We put a nondegenerate symmetric bilinear form on $\mathcal{F}(G)$ via

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) \psi(g)$$

If $\rho: G \to GL(V)$ is a representation, let V^G be the subspace of G-invt vectors, i.e. $V^G = \{v : \rho_g(v) = v \ \forall g \in G\}$

Lemma If $\rho: G \to GL(V)$ is a representation, $\dim V^G = \langle \chi_\rho, \chi_{triv} \rangle$.

Pf We have $P \in \mathrm{End}_G(V)$ given by $P = \frac{1}{|G|} \sum_{g \in G} \rho_g$. Note $P^2 = P$, $\mathrm{Im}\, P = V^G$. So P projects onto V^G. Since $\mathrm{char}\, \mathbb{C} = 0$, $\mathrm{tr}\, P = \dim \mathrm{Im}\, P = \dim V^G$. But also $\mathrm{tr}\, P = \langle \chi_\rho, \chi_{triv} \rangle$ by computation.

Corollary $\dim \mathrm{Hom}_G(V, W) = \langle \chi_\rho, \chi_\sigma \rangle$

Pf $\langle \chi_\rho, \chi_\sigma \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\rho(g^{-1}) \chi_\sigma(g) = \frac{1}{|G|} \sum_{g \in G} \chi_{\rho^* \otimes \sigma}(g) = \langle \chi_{\rho^* \otimes \sigma}, \chi_{triv} \rangle$

Thm Let ρ, σ be irreducible representations.

(a) If $\rho: G \to GL(V)$ and $\sigma: G \to GL(W)$ are not isomorphic, $\langle \chi_\rho, \chi_\sigma \rangle = 0$.

(b) If ρ and σ are equivalent, $\langle \chi_\rho, \chi_\sigma \rangle = 1$.

PF (a) Schur's Lemma $\Rightarrow \mathrm{Hom}_G(V, W) = 0$.

(b) $\langle \chi_\rho, \chi_\sigma \rangle = \dim \mathrm{Hom}_G(V, W) = \dim(\mathbb{C}) = 1$.

Corollary Let $\rho = m_1\rho_1 \oplus \cdots \oplus m_r\rho_r$ be a decomposition into a sum of irreducible representations, where $m_i\rho_i$ is the direct sum of m_i copies of ρ_i. Then $m_i = \dfrac{\langle \chi_\rho, \chi_{\rho_i} \rangle}{\langle \chi_{\rho_i}, \chi_{\rho_i} \rangle}$.

Corollary Two finite-dim'l representations ρ and σ are equivalent $\Leftrightarrow$ their characters coincide.

Corollary A representation ρ is irreducible $\Leftrightarrow \langle \chi_\rho, \chi_\rho \rangle = 1$.

Thm Every irreducible representation ρ of G appears in the regular representation w/ multiplicity $\dim \rho$.

PF $\langle \chi_\rho, \chi_R \rangle = \frac{1}{|G|} \chi_\rho(1) \chi_R(1) = \dim \rho$.

Corollary Let $\rho_1, \ldots, \rho_r$ be the irreducible representations of G and $n_i = \dim \rho_i$. Then $n_1^2 + \cdots + n_r^2 = |G|$.

PF $\dim R = |G| = \chi_R(1) = \sum_{i=1}^{r} n_i \chi_{\rho_i}(1) = \sum_{i=1}^{r} n_i^2$.

The number of representations of a finite group

Defn Let $c(G) = \{ \varphi \in F(G) : \varphi(ghg^{-1}) = \varphi(h) \}$ be the class-functions.

Exercise $\langle , \rangle$ restricts to a nondegenerate

Thm The characters of the irreducible representations on G form an orthonormal basis of $c(G)$.

Pf Wts that if $\varphi \in c(G)$ and $(\varphi, \chi_\rho) = 0$ for any irreducible representation ρ, then $\varphi = 0$.

Claim Let $\rho : G \to GL(V)$ be a representation, $\varphi \in c(G)$ and

$$T = \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) \rho_g$$

Then $T \in \mathrm{End}_G V$ and $\mathrm{tr}\, T = \langle \varphi, \chi_\rho \rangle$.

This is an exercise. Then for ρ irreducible we have that

$$\frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) \rho_g = 0$$

But any representation is a direct sum of irreducibles, so this is true of <u>any</u> representation. In particular for the regular representation R we have

$$\frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) R_g(1) = \frac{1}{|G|} \sum_{g \in G} \varphi(g^{-1}) g = 0$$

Hence $\varphi(g^{-1}) = 0$ for all $g \in G$, i.e. $\varphi = 0$. $\square$