Representation theory of finite groups

Philosophy: Groups arise naturally as symmetries of objects.

- $S_n$ is the group of symmetries of $\{1, \ldots, n\}$
- $P_{2n}$ is the symmetries of a regular $n$-gon

$$P_{12} = \langle r, s : r^6 = 1, s^2 = 1, rs = -sr \rangle$$

- $O(n)$ is the group of distance-preserving maps of $\mathbb{R}^n$ fixing the origin.

Note that this means $O(n)$ is also the group of symmetries of the sphere $S^{n-1}$.
- $SO(n)$ is its orientation-preserving subgroup

Defn:
An action of a group $G$ on a set $X$ is a map $a : G \times X \rightarrow X$

$$(g, x) \mapsto a(g, x)$$

such that $a(ha(g, x)) = a(hg, x)$ and $a(e, x) = x$ (i.e. compatible with group law).

One writes $a(g, x) = gx$ since this is unambiguous (i.e. $h(gx) = (hg)x$).

This is equivalent to specifying a group homomorphism $\alpha : G \rightarrow \text{Perm}(X)$

$g \mapsto \alpha(g)$.

Representation theory starts with the group instead of the object, and asks "on what spaces does $G$ act?"
Generalized actions is an awful problem.

Defn A linear representation \( \rho \) of \( G \) on a vector space over a field \( K \) is an action preserving the linear structure

\[ \rho(g)(v_1 + v_2) = \rho(g)v_1 + \rho(g)v_2 \quad \forall \ v_1, v_2 \in V \]

\[ \rho(g)(kv) = k\rho(g)v \quad \forall \ k \in K, v \in V \]

or equivalently a homomorphism \( \rho: G \to GL(V) \)

\( g \mapsto \rho(g) \)

In what follows, typically \( K = \mathbb{C} \), \( \dim(V) = n < \infty \).

Examples

* \( G \) any group, \( V = \mathbb{C}_1 \), \( \rho: G \to GL(\mathbb{C}) \) is the trivial rep,

  \( g \mapsto \text{Id} \)

* \( G = S_n \), \( V = \mathbb{R}^n \), \( \rho: S_n \to GL(\mathbb{R}) \) is the sign rep,

  \( \sigma \mapsto \text{sgn}(\sigma) \)

* \( G = \mathbb{Z}_1 \), \( V = \mathbb{R}^2 \), \( \rho: \mathbb{Z}_1 \to GL(\mathbb{R}^2) \)

  \( n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \)

* \( G = S_n \), \( V = \mathbb{R}_n \), \( \rho_\sigma: \mathbb{R}_n \to \mathbb{R}_n \) sends \( \rho_\sigma(e_1) = e_{\sigma(1)} \),

  permutation representation
$G = G; \ V = \mathcal{F}(G) = \bigoplus_{x \in G} \mathbb{C} \ G \to \mathbb{K}$ Functions on $G$ w/ values in $\mathbb{K}$

\[ p_{g}(x)(h) = \epsilon(hg) \]

**Exercise** Check this is indeed a representation.

Recall $\mathbb{K}[G]$ the group algebra is the vector space of linear combinations $\sum \epsilon_{g} g$, $g \in \mathbb{K}$, w/ the obvious multiplication making it an algebra.

The regular representation $\pi: G \to GL(\mathbb{K}[G])$ is $\pi_{h} (\sum \epsilon_{g} g) = \sum \epsilon_{hg} g$

**Defn.** Two representations of $G$, $\rho: G \to GL(V)$ and $\sigma: G \to GL(W)$ are equivalent or isomorphic if $\exists$ an invertible linear transformation $T: V \to W$ w/ $T \circ \rho_{g} = \sigma_{g} \circ T \ \forall \ g \in G$

**Example** For $G$ finite, $\pi$ and $\pi$ above are equivalent via

$T: \mathcal{F}(G) \to \mathbb{K}[G]$, so that

\[ T(\rho_{g}(x)) = \sum_{x \in G} \epsilon(x) x^{-1} \]

\[ = \sum_{y \in G} \epsilon(y) y^{-1} \]

\[ = \pi_{g}(T\epsilon) \]

**Remark** A representation of $G$ is equivalently a module over $\mathbb{K}[G]$. 
Ways of producing new representations

Let If $H$ is a subgroup of $G$, then $\rho: G \to GL(V)$ induces a restriction $\rho|_H: H \to GL(W)$, written $\text{Res}_H \rho$.

Lift If $\varphi: G \to H$ is a homomorphism of groups then $\rho \circ \varphi: G \to GL(V)$ is a representation of $G$ for every $\rho: H \to GL(V)$ a representation of $H$.

Most commonly: $N$ is a normal subgroup of $G$ and $H = G/N$, w/ $\pi$ the projection

Direct sum If we have $\rho: G \to GL(V)$ and $\sigma: G \to GL(W)$, then $\rho \otimes \sigma: G \to GL(V \otimes W)$ maps $(\rho \otimes \sigma)_g (v, w) = (\rho_g (v), \sigma_g (w))$

Tensor Product In we have $\rho$ and $\sigma$ as above, $\rho \otimes \sigma: G \to GL(V \otimes W)$ via $(\rho \otimes \sigma)_g (v, w) = \rho_g (v) \otimes \sigma_g (w)$.

Dual Let $V^*$ be the dual space of $V$ and $\langle \cdot, \cdot \rangle: V \otimes V^* \to \mathbb{K}$ be the natural pairing. Given $\rho: G \to GL(V)$ one can define the dual rep $\rho^*: G \to GL(V^*)$ by $\langle \rho_g v, u \rangle = \langle v, \rho_{g^{-1}} u \rangle$

Exercise The regular repn is self-dual for $G$ finite.

Let An operator $T: V \to W$ is an intertwiner if $T \rho_g = \sigma_g T$ for any $g \in G$.

The set of intertwining operators is a vector space $\text{Hom}_G (V, W)$. In the case that $\rho = \sigma$, we have $\text{End}_G (V)$, $\text{Hom}_G (V, V)$ is a $\mathbb{K}$-algebra w/ multiplication given by composition.
Let $\rho: G \to GL(V)$ be a representation. We say $W \subseteq V$ is $G$-invariant if $\rho(g)(w) \in W$ for any $g \in G$.

If $W \subseteq V$ is $G$-invariant, we have the subrepresentation $\rho: G \to GL(W)$ and the quotient representation $\rho: G \to GL(V/W)$.

**Example** IF $\rho: S_n \to GL(k^n)$ is the permutation representation, then

$W = \sum_{(1, \ldots, 1)} x \cdot e_k$

$W' = \sum_{(x_1, \ldots, x_n) : x_1 + \cdots + x_n = 0} x$

**Thm (Mischke)** Let $G$ be a finite group and $\mathfrak{m}$ be its order. Let $\rho: G \to GL(V)$ be a representation and $W \subseteq V$ a $G$-invariant subspace. Then there exists a complementary $G$-invariant subspace, i.e., $W'$ such that $V = W \oplus W'$.

**Proof** Let $W'$ be any subspace such that $W \oplus W' = V$. Let $P$ be the corresponding projection onto $W$ with kernel $W'$ so that $P^2 = P$. Let

$\quad \hat{\rho} := \frac{1}{|G|} \sum_{g \in G} \rho(g)\rho(g)^{-1}$

We see that $\rho(g)\rho(g)^{-1} = \hat{\rho} \quad \forall g \in G$, so $\rho(g)\hat{\rho} = \hat{\rho} \rho(g)$, i.e., $\hat{\rho} \in \text{End}_G(V)$. Moreover $\hat{\rho}|_W = \text{Id}$ and $\text{Im} \hat{\rho} = W$, so $\hat{\rho}^2 = \hat{\rho}$. Let $W' = \ker \hat{\rho}$. Then if $w \in W'$, $\hat{\rho}(\rho(g)w) = \rho(g)\hat{\rho}(w) = \rho(g)(0) = 0$, so $\rho(g)w \in \ker \hat{\rho}' = W'$. Moreover we claim that $V = W \oplus W'$. Indeed $W \cap W' = 0$ since $\hat{\rho}|_W = \text{Id}$. And if $w \in V$, we can write $w = \hat{\rho}v + (v - \hat{\rho}v)$. So we are done.
Note that we used finiteness and \( \frac{1}{|G|} \) making sense (char \( k \neq |G| \)),

\[
\mathbf{R} \rightarrow GL(\mathbf{R}^2) \quad \text{has a single invt subspace, Span } \{ (0,1) \}
\]

In the permutation rep, \( W \) and \( \bar{W} \) are complementary \( (=) \) char \( k + n \).

**Defn** A nonzero representation is called **irreducible** if it does not contain any proper nonzero \( G \)-invariant subspace.

**Exercise** The dimension of an irreducible representation of a finite group \( G \) cannot be larger than \( G \).

**Lemma (Schur)** Let \( p: G \rightarrow GL(V) \) and \( \sigma: G \rightarrow GL(W) \) be two irreducible representations. IF \( T \in \text{Hom}_G(V,W) \), then \( T=0 \) or \( T \) is an isomorphism.

**Pf** We have that \( \ker T \) and \( \text{Im} T \) are \( G \)-invt subspaces of \( V \) and \( W \) respectively. Then by irreducibility of \( p \), \( \ker T = V \) or \( \ker T = 0 \). Likewise \( \text{Im} T = W \) or \( \text{Im} T = 0 \).

**Corollary**

1. IF \( p: G \rightarrow GL(W) \) is irreducible, \( \text{End}_G(V) \) is a division ring.
2. IF \( \text{char} \neq |G| \), \( \text{End}_G(V) \) is a division ring \( \Rightarrow \) \( p \) irreducible.
3. IF \( k \) is algebraically closed and \( p \) is irreducible, \( \text{End}_G(V) = k \).
4. Schur's Lemma
5. IF \( V \) is reducible, \( V = V_1 \oplus V_2 \) for some proper invariant subspaces,
Let \( p_1, p_2 \) be the corresponding projections, Then \( p_2 \circ p_1 = 0 \), so \( \text{End}_G(V) \) has zero divisors.
6. Consider \( T \in \text{End}_G(V) \). Then \( T \) has some eigenvalue \( \lambda \) and \( T - \lambda I \in \text{End}_G(V) \). \( T - \lambda I \) is certainly not invertible, so \( \lambda = 0 \) or \( T = 0 \).
A representation is called **completely reducible** if it splits into a direct sum of irreducible subrepresentations.

**Lemma** Let \( p : G \to GL(V) \) be a representation of \( G \). The following are equivalent:

1. \( p \) is completely reducible.
2. For any \( G \)-invariant subspace \( V' \subseteq V \), there exists a \( G \)-invariant subspace \( V'' \subseteq V \) such that \( V = V' \oplus V'' \).

**Proof** For \( V \) finite-dimensional, this is easy. For infinite-dimensional one has to use Zorn's lemma (the proof is in Gruson-Serganova).

**Corollary** Let \( G \) be a finite group and let \( k \) be a field such that \( \text{char } k + 1 \mid |G| \). Then every representation of \( G \) is completely reducible.

### Characters

**Definition** For a finite-dimensional repn \( p : G \to GL(V) \) the function \( \chi_p : G \to k \) \( g \mapsto \text{tr}(p(g)) \) is called the **character** of the representation.

We easily see that:

1. \( \chi_p(1) = \dim_p \)
2. If \( p \cong \sigma \), \( \chi_p = \chi_\sigma \)
3. \( \chi_{p \oplus \sigma} = \chi_p \pm \chi_\sigma \)
4. \( \chi_{p \otimes \sigma} = \chi_p \chi_\sigma \)
5. \( \chi_p(g^{-1}) = \chi_p(g)^{-1} \)
6. \( \chi_p(ghg^{-1}) = \chi_p(h) \)
Example For the regular representation $R: \mathfrak{g}[\mathfrak{g}]$, $x_{R}(g) = 0$ for $g \neq 1$ and $x_{R}(1) = |\mathfrak{g}|$.

Lemma If $\mathfrak{h} = \mathfrak{g}$ and $G$ finite, for any finite dimensional $\mathfrak{g}$ and $g \in G$ we have $x_{\mathfrak{g}}(g) = x_{\mathfrak{g}}(g^{-1})$.

Proof We have $x_{\mathfrak{g}}(g)$ is the sum of the eigenvalues of $g$. Since $g$ is a root of $1$ and has finite order every eigenvalue is a root of $1$. Thus the eigenvalues of $g^{-1}$ are the complex conjugates.

Orthogonality Let $G$ be finite, $\mathfrak{h} = \mathfrak{g}$.

We put a nondegenerate symmetric bilinear form on $F(\mathfrak{g})$ via

$$\langle e_{j}, u \rangle = \frac{1}{|G|} \sum_{g \in G} e_{j}(g^{-1}) u(g)$$

IF $\rho: G \rightarrow GL(V)$ is a representation, let $V_{G}$ be the subspace of $G$-invariant vectors, i.e., $V_{G} = \{v \in V: \rho_{g}(v) = v, \forall g \in G\}$

Lemma If $\rho: G \rightarrow GL(V)$ is a representation, $\dim V_{G} = \langle x_{\mathfrak{g}}, x_{\text{ev}} \rangle$.

Proof We have $P \in \text{End}_{G}(V)$ given by $P = \frac{1}{|G|} \sum_{g \in G} \rho_{g}$. Note $P^{2} = P$, $\text{Im} P = V_{G}$. So $P$ projects onto $V_{G}$. Since $\text{char} G = 0$, $tr P = \dim \text{Im} P = \dim V_{G}$, but also $tr P = \langle x_{\mathfrak{g}}, x_{\text{ev}} \rangle$ by computation.

Corollary $\dim \text{Hom}_{G}(V, W) = \langle x_{\mathfrak{g}}, x_{\sigma} \rangle$.

Proof $\langle x_{\mathfrak{g}}, x_{\sigma} \rangle = \frac{1}{|G|} \sum_{g \in G} x_{\mathfrak{g}}(g^{-1}) x_{\sigma}(g) = \frac{1}{|G|} \sum_{g \in G} x_{\rho \circ \sigma}(g) = \langle x_{\rho \circ \sigma}, x_{\text{ev}} \rangle$.
Thm Let $p, \sigma$ be irreducible representations.

(a) If $p : G \to GL(V)$ and $\sigma : G \to GL(W)$ are not isomorphic, $\langle x_p, x_\sigma \rangle = 0$.

(b) If $p$ and $\sigma$ are equivalent, $\langle x_p, x_\sigma \rangle = 1$.

Proof by Schur's Lemma $\Rightarrow$ $\text{Hom}_G(V, W) = 0$.

$\langle x_p, x_\sigma \rangle = \dim \text{Hom}_G(V, W) = \dim(\sigma) = 1$.

Corollary Let $p = m_1 p_1 \oplus \cdots \oplus m_r p_r$ be a decomposition into a sum of irreducible representations, where $m_i p_i$ is the direct sum of $m_i$ copies of $p_i$. Then $m_i = \frac{\langle x_p, x_{p_i} \rangle}{\langle x_{p_i}, x_{p_i} \rangle}$.

Corollary Two finite-dimensional representations $p$ and $\sigma$ are equivalent (\iff) their characters coincide.

Corollary A representation $p$ is irreducible (\iff) $\langle x_p, x_p \rangle = 1$.

Thm Every irreducible representation $p$ of $G$ appears in the regular representation with multiplicity $\dim p$.

Proof $\langle x_p, x_{\text{reg}} \rangle = \frac{1}{|G|} \chi_p(1) \chi_{\text{reg}}(1) = \dim p$.

Corollary Let $p_1, \ldots, p_r$ be the irreducible representations of $G$ and $n_i = \dim p_i$.

Then $n_1^2 + \cdots + n_r^2 = |G|$.

Proof $\dim \text{reg} = |G| \Rightarrow \chi_{\text{reg}}(1) = \sum_{i=1}^r n_i \chi_{p_i}(1) = \sum_{i=1}^r n_i^2$. 

The number of representations of a finite group

**Def.** Let \( C(G) = \{ \xi \in \mathbb{F}(G) : \xi(ghg^{-1}) = \xi(h) \} \) be the class functions.

**Exercise.** \( \langle , \rangle \) restricts to a nondegenerate

**Thm.** The characters of the irreducible representations on \( G \) form an orthonormal basis of \( C(G) \).

**PF:** We see that if \( \xi \in C(G) \) and \( \langle \xi, \chi_p \rangle = 0 \) for any irreducible representation \( \chi_p \), then \( \xi = 0 \).

**Claim.** Let \( p: G \rightarrow GL(V) \) be a representation, \( \xi \in C(G) \) and

\[
T = \frac{1}{|G|} \sum_{g \in G} \xi(g^{-1}) p_g
\]

Then \( T \in \text{End}_G V \) and \( \text{tr} T = \langle \xi, \chi_p \rangle \).

This is an exercise. Then for \( p \) irreducible we have that

\[
\frac{1}{|G|} \sum_{g \in G} \xi(g^{-1}) p_g = 0
\]

But any representation is a direct sum of irreducibles, so this is true of any representation. In particular for the regular representation \( R \) we have

\[
\frac{1}{|G|} \sum_{g \in G} \xi(g^{-1}) p_g(1) = \frac{1}{|G|} \sum_{g \in G} \xi(g^{-1}) g = 0
\]

Hence \( \xi(g^{-1}) = 0 \) for all \( g \in G \), i.e. \( \xi = 0 \). \( \Box \)