Math 540: Exercises for Week 3

Reading: Hatcher Section 1.1-2

1. Hatcher 1.1.1
2. Hatcher 1.1.5
3. Hatcher 1.1.7
4. Hatcher 1.1.9
5. Hatcher 1.1.10
6. Let G be a path-connected topological group. Let $f\colon I \to G$ and $g\colon I \to G$ be two loops in G based at the identity. Observe that there are two natural notions of the product of f and g: the composition $f \cdot g$ defined in class, and the pointwise product $fg(s) = f(s)g(s)$. Prove that these notions agree up to homotopy of loops. Use this to show the fundamental group of a topological group is always abelian.
7. Let $H^1(X) = [X, S^1]$ be the set of homotopy classes of continuous maps from X to the circle. (No basepoints involved.)
   (a) Remember that S^1 has the structure of a topological group. Use this to give $H^1(X)$ the structure of an abelian group.
   (b) What is $H^1(\{\text{pt}\})$?
   (c) What is $H^1(S^1)$? (Use what you know about $\pi_1(S^1)$.)
   (d) Show that H^1 is functorial in the following sense: Given $f\colon X \to Y$ a continuous map, there is an induced map $f^*\colon H^1(Y) \to H^1(X)$. Moreover if $g\colon Y \to Z$ then $(g \circ f)^* = f^* \circ g^*$ as maps $H^1(Z) \to H^1(X)$.
   (e) Show that if f and h are homotopic as maps $X \to Y$ then $f^* = h^*$. Conclude that if $X \simeq Y$ then $H^1(X) \simeq H^1(Y)$.
   (f) Use H^1 to prove there is no retraction $D^2 \to S^1$, giving yet another proof of the Brouwer fixed point theorem.