Math 540: Exercises for Week 3

Reading: Hatcher Section 1.1-2

1. Hatcher 1.1.1
2. Hatcher 1.1.5
3. Hatcher 1.1.7
4. Hatcher 1.1.9
5. Hatcher 1.1.10

6. Let $G$ be a path-connected topological group. Let $f: I \to G$ and $g: I \to G$ be two loops in $G$ based at the identity. Observe that there are two natural notions of the product of $f$ and $g$: the composition $f \cdot g$ defined in class, and the pointwise product $fg(s) = f(s)g(s)$. Prove that these notions agree up to homotopy of loops. Use this to show the fundamental group of a topological group is always abelian.

7. Let $H^1(X) = [X,S^1]$ be the set of homotopy classes of continuous maps from $X$ to the circle. (No basepoints involved.)

(a) Remember that $S^1$ has the structure of a topological group. Use this to give $H^1(X)$ the structure of an abelian group.

(b) What is $H^1(\{\text{pt}\})$?

(c) What is $H^1(S^1)$? (Use what you know about $\pi_1(S^1)$.)

(d) Show that $H^1$ is functorial in the following sense: Given $f: X \to Y$ a continuous map, there is an induced map $f^*: H^1(Y) \to H^1(X)$. Moreover if $g: Y \to Z$ then $(g \circ f)^* = f^* \circ g^*$ as maps $H^1(Z) \to H^1(X)$.

(e) Show that if $f$ and $h$ are homotopic as maps $X \to Y$ then $f^* = h^*$. Conclude that if $X \simeq Y$ then $H^1(X) \simeq H^1(Y)$.

(f) Use $H^1$ to prove there is no retraction $D^2 \to S^1$, giving yet another proof of the Brouwer fixed point theorem.