Math 354, Section 04
Linear Optimization

Quiz

Instructions: You have 40 minutes to complete the quiz. There are six questions, worth a total of eighteeen points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: _____

RUID: _____

Question	Points	Score
1	3	
2	3	
3	3	
4	3	
5	3	
6	3	
Total:	18	

Consider the following linear programming problem.

A doctor treats a common minor ailment with the commercially available pills A and B. Pill A costs 40 cents per pill and Pill B costs 10 cents per pill. They each contain a mixture of compound C, the actual medication needed, and an activator, compound D. Pill A contains 4 mg of Compound C and Pill B contains 2 mg of Compound C; they each contain .5 mg of the activator Compound D. The usual dosage needed is 10 mg of Compound C per day; furthermore, the activator can be harmful in large doses, and the doctor therefore limits the amount of Compound D consumed to no more than 2 mg per day. Dosages may if necessary be in half-pill units using a pill-cutter. What is the cheapest prescription the doctor can write that contains enough Compound C and a safe amount of Compound D?

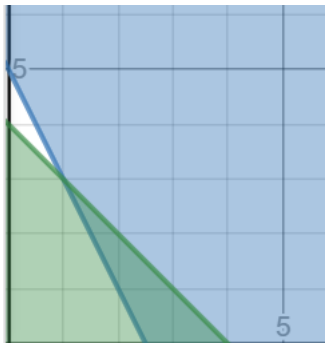
1. [3pts.] Write down this problem as a system of equations in standard form, including both the objective functions and the constraints.

Solution: Let x_1 be the number of pill A and x_2 be the number of pill B in the prescription. We wish to maximize $z = -40x_1 - 10x_2$ subject to the constraints

$$\begin{cases} -4x_1 - 2x_2 \leq -10 \\ .5x_1 + .5x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{cases}$$

2. [3pts.] Graph the set of feasible solutions to the problem.

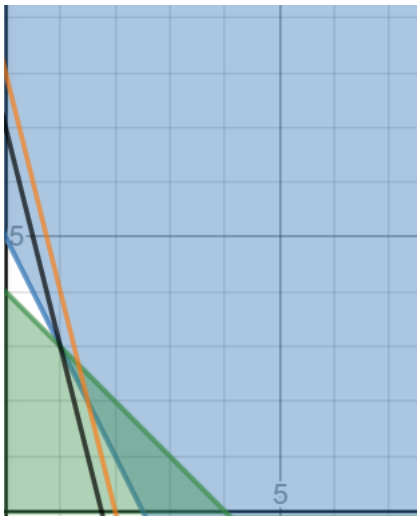
Solution: The set of feasible solutions is the doubly-shaded region shown.



Citation: Image from Desmos.

3. [3pts.] Find the optimal prescription. Justify your answer.

Solution: We draw some level sets of $z = -40x_1 - 10x_2$ and conclude that the optimal prescription is at the intersection point $(1, 3)$: one of Pill A and three of Pill B, costing 70 cents per day.



4. [3pts.] Transform your set of equations from question (2) into canonical form. Write this down both as a system of equations and in matrix notation.

Solution: We want to maximize $z = -40x_1 - 10x_2$ subject to the constraints

$$\begin{cases} -4x_1 - 2x_2 + u_1 = -10 \\ .5x_1 + .5x_2 + u_2 = 2 \\ x_1, x_2, u_1, u_2 \geq 0 \end{cases}$$

Equivalently in matrix form we would like to maximize

$$z = \begin{bmatrix} -40 & -10 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix}$$

subject to the constraints

$$\begin{bmatrix} -4 & -2 & 1 & 0 \\ .5 & .5 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -10 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

5. [3pts.] What are the values of the slack variables from question (4) at the optimal solution you found in question (3)? What does that represent physically?

Solution: At the optimal solution $u_1 = u_2 = 0$, meaning that the prescription provides exactly the necessary 10 mg of compound C and is at exactly the safe limit of Compound D.

6. [3pts.] Suppose the price of Pill B rises to 25 cents per pill. Does that change what the cheapest prescription is?

Solution: Yes, as we see from the level sets of $z = -40x_1 + -25x_2$, this makes it cheaper to give 2.5 pills of type A instead. This costs \$1 per day.