**Math 354, Section 04**
**Linear Optimization**

**Midterm 2**

**Instructions:** You have 80 minutes to complete the exam. There are four questions, worth a total of 40 points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 10 | |
| 3 | 10 | |
| Total: | 40 | |

1. Consider the following linear programming problem.

   A lawn products company has available 100 tons of nitrate and 60 tons of phosphate to use in producing its three types of fertilizer for sale this week. Their three fertilizer types are regular lawn, super lawn, and garden. Every thousand bags of regular lawn fertilizer requires 4 tons of nitrate and 2 tons of phosphate, and can be sold for a profit of \$300. Every thousand bags of super lawn fertilizer require 3 tons of nitrate and 3 tons of phosphate, and can be sold for a profit of \$500. Every thousand bags of garden fertilizer requires 3 tons of nitrate and 2 tons of phosphate and can be sold for a profit of \$400. Fractional numbers of bags of fertilizer are ok and can be retailed appropriately.

   (a) [2pts.] Write this situation down as a linear programming problem in standard form.

   **Solution:** Let $x_1$ be regular lawn fertilizer, $x_2$ be super lawn fertilizer, and $x_3$ be garden fertilizer. We wish to maximize $z = 300x_1 + 500x_2 + 400x_3$ subject to the constraints

$$\begin{cases} 4x_1 + 3x_2 + 3x_3 \leq 100 \\ 2x_1 + 3x_2 + 2x_3 \leq 60 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

   (b) [5pts.] Solve the linear programming problem using the simplex method.

   **Solution:** We start by adding two slack variables to the constraints

$$\begin{cases} 4x_1 + 3x_2 + 3x_3 + u_1 = 100 \\ 2x_1 + 3x_2 + 2x_3 + u_2 = 60 \\ x_1, x_2, x_3, u_1, u_2 \geq 0 \end{cases}$$

   Now we can form an initial tableau as follows.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 4 | 3 | 3 | 1 | 0 | 0 | 100 |
| $u_2$ | 2 | 3 | 2 | 0 | 1 | 0 | 60 |
| | $-300$ | $-500$ | $-400$ | 0 | 0 | 1 | 0 |

   We let $x_2$ be the entering variable, which makes $u_2$ the departing variable. We pivot accordingly.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 2 | 0 | 1 | 1 | $-1$ | 0 | 40 |
| $x_2$ | $2/3$ | 1 | $2/3$ | 0 | $1/3$ | 0 | 20 |
| | $100/3$ | 0 | $-200/3$ | 0 | $500/3$ | 1 | 10000 |

Our new entering variable is $x_3$ and the departing variable is $x_2$. We pivot.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | $-3/2$ | 0 | 1 | $-3/2$ | 0 | 10 |
| $x_3$ | 1 | $3/2$ | 1 | 0 | $1/2$ | 0 | 30 |
| | 100 | 100 | 0 | 0 | 200 | 1 | 12000 |

Now we are done. The company should make 30 thousand bags of garden fertilizer for a max profit of $12,000.

(c) [2pts.] The lawn company has the option of buying another ton of nitrate to increase production this week. Should it do that? If so, what is the maximum price it should pay for the nitrate?

**Solution:** No, the marginal value of a ton of nitrate to the company, which is the value under $u_1$ in the objective row of the tableau, is zero.

(d) [2pts.] The lawn company has the option of selling off a ton of phosphate this week. What is the minimum price it should accept for the phosphate?

**Solution:** The marginal value of a ton of phosphate to the company is the value of $u_2$ in the objective row, namely 200, so the company should not accept any less than $200 for the phosphate.

(e) [3pts.] The lawn products company expects that next week, because most people will have planted their gardens already, it will have to sell garden fertilizer at a discount. By how much can it lower the price of the garden fertilizer per thousand bags such that the solution you found in part (b) remains optimal?

**Solution:** We are interested in what happens if we change $c_3$ to $c_3' = c_3 + \Delta c_3$. This would have the following effect on the final tableau above.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | $-3/2$ | 0 | 1 | $-3/2$ | 0 | 10 |
| $x_3$ | 1 | $3/2$ | 1 | 0 | $1/2$ | 0 | 30 |
| | 100 | 100 | $-\Delta c_3$ | 0 | 200 | 1 | 12000 |

Clearing out the objective row we obtain

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | $-3/2$ | 0 | 1 | $-3/2$ | 0 | 10 |
| $x_3$ | 1 | $3/2$ | 1 | 0 | $1/2$ | 0 | 30 |
| | $100 + \Delta c_3$ | $100 + 3/2\Delta c_3$ | 0 | 0 | $200 + 1/2\Delta c_3$ | 1 | 12000 |

This is still optimal if all the values in the objective row are nonnegative, which

we see is true if $\Delta c_3 \geq -200/3$. So the price of garden fertilizer could fall by $\sim$ \$66.67 per thousand bags before this solution is no longer optimal.

(f) [3pts.] The lawn products company expects that next week many people will want to encourage their lawns to grow, and the price at which it could sell super lawn fertilizer may increase. How much does it need to increase before the company's best strategy changes?

**Solution:** We are interested in what happens when we replace $c_2 = 500$ with $c_2' = c_2 + \Delta c_2$. This has the following effect on the final tableau above.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | $-3/2$ | 0 | 1 | $-3/2$ | 0 | 10 |
| $x_3$ | 1 | $3/2$ | 1 | 0 | $1/2$ | 0 | 30 |
| | 100 | $100 - \Delta c_2$ | 0 | 0 | 200 | 1 | 12000 |

We see this tableau stops being optimal once the price rises by \$100 per thousand bags, to \$600 per thousand bags. At that point the company should make some super lawn fertilizer.

(g) [3pts.] Due to a miscommunication only 80 tons of nitrate have been delivered to the company factory! What should the company do under this circumstance, and what is the new maximum profit for the week?

**Solution:** This has the effect of adding $\Delta b_1 = -20$ times the slack column $u_1$ corresponding to the nitrate to the final column. We get the infeasible tableau below.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | $-3/2$ | 0 | 1 | $-3/2$ | 0 | $-10$ |
| $x_3$ | 1 | $3/2$ | 1 | 0 | $1/2$ | 0 | 30 |
| | 100 | 100 | 0 | 0 | 200 | 1 | 12000 |

We must do a dual pivot to restore feasibility. We let $u_1$ be the departing variable and $x_2$ be the entering variable, and pivot.

| | $x_1$ | $x_2$ | $x_3$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|
| $x_2$ | $-2/3$ | 1 | 0 | $-2/3$ | 1 | 0 | $20/3$ |
| $x_3$ | 2 | 0 | 1 | 1 | $-1$ | 0 | 20 |
| | $500/3$ | 0 | 0 | $200/3$ | 100 | 1 | $34000/3$ |

So the company should make $20/3$ thousand bags of super lawn fertilizer and 20 thousand bags of garden fertilizer for a total profit of \$34000 $\sim$ \$11,333.33.

2. Consider the following linear programming problem: maximize $z = 3x_1 - 2x_2 + x_4$ subject

to

$$\begin{cases} x_1 + x_3 + x_4 \leq 10 \\ 2x_2 - 6x_3 \geq 8 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

(a) [5pts.] Solve the problem using the two-phase method.

**Solution:** We add slack and artificial variables as follows.

$$\begin{cases} x_1 + x_3 + x_4 + u_1 = 10 \\ 2x_2 - 6x_3 - u_2 + y_1 = 8 \\ x_1, x_2, x_3, x_4, u_1, u_2, y_1 \geq 0 \end{cases}$$

For Phase I, we wish to maximize $z = -y_1$. We start with

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $u_1$ | $u_2$ | $y_1$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 10 |
| $y_1$ | 0 | 2 | $-6$ | 0 | 0 | $-1$ | 1 | 0 | 8 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |

Note that using $x_1$ as the initial basic variable in the first row would also have been fine (and slightly quicker). We clear out the objective row to get our intial tableau:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $u_1$ | $u_2$ | $y_1$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 10 |
| $y_1$ | 0 | 2 | $-6$ | 0 | 0 | $-1$ | 1 | 0 | 8 |
| | 0 | $-2$ | 6 | 0 | 0 | 1 | 0 | 1 | $-8$ |

We have an entering variable of $x_2$ and a departing variable of $y_1$. We pivot accordingly.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $u_1$ | $u_2$ | $y_1$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 10 |
| $x_2$ | 0 | 1 | $-3$ | 0 | 0 | $-1/2$ | $1/2$ | 0 | 4 |
| | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |

Phase I is complete. Now we replace the objective row with the original objective function. We don't drop the $y_1$ column, because we can see part (b) coming.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $u_1$ | $u_2$ | $y_1$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 10 |
| $x_2$ | 0 | 1 | $-3$ | 0 | 0 | $-1/2$ | $1/2$ | 0 | 4 |
| | $-3$ | 2 | 0 | $-1$ | 0 | 0 | 0 | 1 | 0 |

We clear out the objective row to get an initial tableau for Phase II:

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $u_1$ | $u_2$ | $y_1$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|
| $u_1$ | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 10 |
| $x_2$ | 0 | 1 | $-3$ | 0 | 0 | $-1/2$ | $1/2$ | 0 | 4 |
| | $-3$ | 0 | 6 | $-1$ | 0 | 1 | $-1$ | 1 | $-8$ |

We now let $x_1$ be the entering variable, which makes $u_1$ the departing variable. We pivot accordingly.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $u_1$ | $u_2$ | $y_1$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 10 |
| $x_2$ | 0 | 1 | $-3$ | 0 | 0 | $-1/2$ | $1/2$ | 0 | 4 |
| | 0 | 0 | 9 | 2 | 3 | 1 | $-1$ | 1 | 22 |

So the optimal value is 22 and the optimal solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 0 \\ 0 \end{bmatrix}.$$

(b) [5pts.] What is the dual of this linear programming problem? What are the optimal solution and optimal value of the dual problem?

**Solution:** We put the original problem in standard form, in particular changing the constraints to

$$\begin{cases} x_1 + x_3 + x_4 \leq 10 \\ -2x_2 + 6x_3 \leq -8 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

Then we see that the dual problem is to minimize $z' = 10w_1 - 8w_2$ subject to the constraints

$$\begin{cases} w_1 \geq 3 \\ -2w_2 \geq -2 \\ w_1 + 6w_2 \geq 0 \\ w_1 \geq 1 \\ w_1, w_2 \geq 0 \end{cases}$$

The optimal value is the same as the primal problem, namely 22. Looking at the entries of the objective row under the initial basic variables in the final tableau above we conclude that it occurs at the optimal solution

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

keeping in mind that we need to switch the sign of the value under the artificial variable for an inequality that started out greater than or equal to a positive number.

3. Consider the following simplex tableaux.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x_5$ | 2 | −2 | 0 | 0 | 1 | 0 | 0 | 0 | 2 |
| $x_3$ | −3 | 3 | 1 | 0 | 0 | 0 | 2 | 0 | 0 |
| $u_1$ | −4 | 0 | 0 | 0 | 0 | 1 | 4 | 0 | 7 |
| $x_4$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 3 |
| | −1 | 3 | 0 | 0 | 0 | 0 | −7 | 1 | −5 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $u_1$ | $u_2$ | $z$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x_5$ | −2 | −2 | 0 | 0 | 1 | 0 | 0 | 0 | −12 |
| $x_3$ | −3 | 3 | 1 | 0 | 0 | 0 | 2 | 0 | 2 |
| $u_1$ | −4 | 0 | 0 | 0 | 0 | 1 | −2 | 0 | −9 |
| $x_4$ | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 3 |
| | 4 | 3 | 0 | 0 | 0 | 0 | 7 | 1 | 62 |

(a) [2pts.] For the first tableau, what basic feasible solution does the tableau represent?

**Solution:** The tableau corresponds to the basic feasible solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 2 \\ 7 \\ 0 \end{bmatrix}.$$

(b) [2pts.] For the first tableau, what is the correct entering variable if the standard rule for the simplex method is used, and in that case what is the departing variable?

**Solution:** Using the standard rule we look for the most negative entry in the objective row, and conclude that the entering variable is $u_2$. Then the $\theta$-ratios corresponding to the positive entries in the pivotal column are 0 for $x_3$ and 7/4 for $u_1$, so we let $x_3$ be the departing variable.

(c) [3pts.] Suppose that my attempt to run the simplex rule using the standard rule on the first tableau seems to be cycling. To fix this, what choice of entering and departing variable should I make?

**Solution:** We should use Bland's Rule. We let the leftmost negative entry in the objective row determine the departing variable, namely $x_1$. Then the $\theta$-

ratios corresponding to the positive entries in the pivotal column are 1 for $x_5$ and 3 for $x_4$, so we let $x_5$ be the departing variable.

(d) [3pts.] For the second tableau, suppose I want to do a dual pivot toward making the tableau feasible. What entering and departing variables should I use?

**Solution:** We let $x_5$ be the departing variable since it has the most negative entry in the objective row. Then for the two negative entries in the pivotal row, the value of the entry in the objective row divided by the entry in the pivotal row is $-2$ for $x_1$ and $-3/2$ for $x_2$. Therefore we choose $x_2$ to be the departing variable.