Math 354, Section 04
Linear Optimization

Sample Midterm 2

Instructions: You have 80 minutes to complete the exam. There are three questions, worth a total of 40 points. Partial credit will be given for progress toward correct solutions where relevant. Solutions will be graded on their correctness relative to previous steps where this makes sense. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

Question	Points	Score
1	20	
2	10	
3	10	
Total:	40	

1. Consider the following linear programming problem.

   A farmer owns a farm that produces corn, soybeans, and oats. She has 12 acres of land available for cultivation. Planting requires different amounts of money for seed and different amounts of total labor. (After planting the requirements for cultivating and harvesting the different crops are sufficiently similar that they can be ignored for purposes of optimization.) Corn costs a total of \$36 per acre to plant and needs 6 hours of labor per acre; the farmer expects corn to yield a net profit of \$40 per acre. Soybeans cost a total of \$24 per acre to plant and need 6 hours of labor per acre; the farmer expects soybeans to yield a net profit of \$30 per acre. Oats cost \$18 per acre to plant and need 2 hours of labor per acre; the farmer expects oats to yield a net profit of \$20 per acre. The farmer has 48 total labor-hours available for planting and \$360 available for initial costs. How much of each crop should be planted to maximize profit, and what is the maximum profit thusly obtained?

   (a) [2pts.] Write down the situation above as a linear programming problem in standard form.

   Solution: Let x_1 be the acres of corn planted, x_2 be the acres of soybeans planted, and x_3 be the acres of oats planted. Then the problem is to maximize $z = 40x_1 + 30x_2 + 20x_3$ subject to the constraints

$$\begin{cases} x_1 + x_2 + x_3 \leq 12 \\ 6x_1 + 6x_2 + 2x_3 \leq 48 \\ 36x_1 + 24x_2 + 18x_3 \leq 360 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

   (b) [6pts.] Solve this problem using the simplex method, and determine the optimal solution and optimal value.

   Solution: We put this problem into canonical form by adding slack variables as follows.

$$\begin{cases} x_1 + x_2 + x_3 + u_1 = 12 \\ 6x_1 + 6x_2 + 2x_3 + u_2 = 48 \\ 36x_1 + 24x_2 + 18x_3 + u_3 = 360 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

   This gives us an intial tableau of the following form

	x_1	x_2	x_3	u_1	u_2	u_3	z	
u_1	1	1	1	1	0	0	0	12
u_2	6	6	2	0	1	0	0	48
u_3	36	24	18	0	0	1	0	360
	-40	-30	-20	0	0	0	1	0

We let x_1 be the entering variable, which gives us θ-ratios of $\{12, 8, 10\}$ in descending order and makes the departing variable u_2. We pivot to obtain the following.

	x_1	x_2	x_3	u_1	u_2	u_3	z	
u_1	0	0	$2/3$	1	$-1/6$	0	0	4
x_1	1	1	$1/3$	0	$1/6$	0	0	8
u_3	0	-12	6	0	-6	1	0	72
	0	10	$-20/3$	0	$20/3$	0	1	320

Next we let x_3 be the entering variable, which gives us θ-ratios of $\{6, 24, 12\}$ in descending order. So the correct departing variable is u_1. We therefore pivot to obtain the following.

	x_1	x_2	x_3	u_1	u_2	u_3	z	
x_3	0	0	1	$3/2$	$-1/4$	0	0	6
x_1	1	1	0	$-1/2$	$1/4$	0	0	6
u_3	0	-12	0	-9	$-9/2$	1	0	36
	0	10	0	10	5	0	1	360

As there are no more negative entries on the objective row, we are done. We conclude that the optimal solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix}$$

so the farmer should plant six acres of corn and six acres of oats for a total profit of \$360.

(c) [5pts.] What is the dual problem to the primal problem from part (a)? What are its optimal solution and optimal value? What do the entries of the optimal solution mean?

Solution: The dual problem is to minimize $z' = 12w_1 + 48w_2 + 360w_3$ subject to the constraints

$$\begin{cases} w_1 + 6w_2 + 36w_3 \geq 40 \\ w_1 + 6w_2 + 24w_3 \geq 30 \\ w_1 + 2w_2 + 18w_3 \geq 20 \\ w_1, w_2, w_3 \geq 0 \end{cases}$$

The optimal value of the dual problem is the same as the optimal value of the primal problem, in this case 360. The optimal solution can be read off the tableau by looking at the entries in the objective row underneath the initial

basic variables and is

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}.$$

These numbers are the marginal values of the three resources (land, labor-hours, and initial capital) to us at the optimal solution; so, we would pay \$10 for an additional acre of land, \$5 for another hour of labor, and wouldn't pay anything for additional funds to spend on planting materials. (If you're wondering what it would mean to pay for money, this can be interpreted as taking out a small loan and paying it back at the end of the season, and the marginal value is then the interest we'd be willing to commit to paying.)

(d) [2pts.] The farmer becomes worried that she might be wrong about what the net profit on soybeans will be at harvest time. What is the range of values the profit on soybeans could be such that the solution you found in part (b) is still optimal?

Solution: Let the initial profit on soybeans be $c_2 = 30$ and suppose we replace it by $c_2' = 30 + \Delta c_2$. This has the effect of changing the entry in the objective row under x_2 to $10 - \Delta c_2$. So, the profit on soybeans can rise by as much as \$10 per acre, to \$40 per acre, before the farmer would want to change her planting strategy.

(e) [2pts.] The farmer becomes worried that she might be wrong about the net profit on corn will be at harvest time. What is the range of values the profit on corn could be such that the solution you found in part (b) is still optimal?

Solution: Let the initial profit on soybeans be $c_1 = 40$ and suppose we replace it by $c_1' = 40 + \Delta c_1$. This has the effect of changing the entry in the objective row under x_1 to $-\Delta c_1$. We then must clear out the objective row by adding Δc_1 times the x_1 row to the objective row, and consider what values of Δc_1 would make one of the entries on the objective row negative. This gives us the equations

$$10 + \Delta c_1 \geq 0 \qquad 10 - 1/2\Delta c_1 \geq 0 \qquad 5 + 1/4\Delta c_1 \geq 0$$

which reduces to $-10 \leq \Delta c_1 \leq 20$. So, the net profit on corn can fall to \$30 per acre or alternately rise to \$60 per acre before the farmer should change her planting strategy.

(f) [3pts.] Unfortunately the farmer turns out to only have 42 hours of labor available! What should she do in this situation to maximize the profit she can get from her farm, and what is her new maximum profit?

Solution: Labor was our second constraint, and the change in the amount available is $\Delta b_2 = -6$, so we add -6 times the column under u_2 to the final column to obtain the new tableau

	x_1	x_2	x_3	u_1	u_2	u_3	z	
x_3	0	0	1	3/2	$-1/4$	0	0	15/2
x_1	1	1	0	$-1/2$	1/4	0	0	9/2
u_3	0	-12	0	-9	$-9/2$	1	0	63
	0	10	0	10	5	0	1	330

This is still feasible (if it wasn't, we would do some dual pivots to restore feasibility). So we conclude that the farmer should plant 4.5 acres of corn and 7.5 acres of oats for a maximum profit of \$330.

2. Consider the following linear programming problem: Maximize $z = 3x_1 - 2x_2$ subject to

$$\begin{cases} x_1 + x_2 + 2x_3 \leq 7 \\ 2x_1 + x_2 + x_3 = 4 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

(a) [5pts.] Solve the problem using the two-phase method.

Solution: We convert add slack and artificial variables to obtain a basis as follows.

$$\begin{cases} x_1 + x_2 + 2x_3 + u_1 = 7 \\ 2x_1 + x_2 + x_3 + y_1 = 4 \\ x_1, x_2, x_3, u_1, y_1 \geq 0 \end{cases}$$

We want to start with the auxiliary problem of maximizing $z = -y_1$, for which we have the following.

	x_1	x_2	x_3	u_1	y_1	z	
u_1	1	1	2	1	0	0	7
y_1	2	1	1	0	1	0	4
	0	0	0	0	1	1	0

We must clear out the objective row to obtain an initial tableau, as follows.

	x_1	x_2	x_3	u_1	y_1	z	
u_1	1	1	2	1	0	0	7
y_1	2	1	1	0	1	0	4
	-2	-1	-1	0	0	1	-4

We choose x_1 for the entering variable, which makes the departing variable y_1. We pivot accordingly.

	x_1	x_2	x_3	u_1	y_1	z	
u_1	0	1/2	3/2	1	−1/2	0	5
x_1	1	1/2	1/2	0	1/2	0	2
	0	0	0	0	1	1	0

Phase I is done. We now restore the original objective function. We don't drop the artificial variable column, because we've noticed part (b) of this problem, but we do ignore that column for the optimality criterion henceforth.

	x_1	x_2	x_3	u_1	y_1	z	
u_1	0	1/2	3/2	1	−1/2	0	5
x_1	1	1/2	1/2	0	1/2	0	2
	−3	2	0	0	0	1	0

We clear out the objective row to get an initial tableau for Phase II.

	x_1	x_2	x_3	u_1	y_1	z	
u_1	0	1/2	3/2	1	−1/2	0	5
x_1	1	1/2	1/2	0	1/2	0	2
	0	7/2	3/2	0	3/2	1	6

We see that this tableau is already optimal, so we are done. The optimal solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

and the optimal value is 6.

(b) [5pts.] Write down the dual to the problem above. What is its optimal solution and optimal value?

Solution: The dual problem is to minimize $z' = 7w_1 + 4w_2$ subject to

$$\begin{cases} w_1 + 2w_2 \geq 3 \\ w_1 + w_2 \geq -2 \\ 2w_1 + w_2 \geq 0 \\ w_1 \geq 0 \end{cases}$$

Note lack of restriction on w_2, which can in principle take positive or negative values. The optimal value is the same as for the primal problem, namely 6, and

from examining the entries in the objective row of the optimal tableau for the primal problem under the initial basic variables we conclude that the optimal solution is

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}.$$

3. Consider the following simplex tableaux.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_4	0	0	2	1	5/2	0	0	0	5
x_1	1	0	5	0	−3	0	−2	0	3
x_6	0	0	3	0	4	1	−4	0	1
x_2	0	1	0	0	3/2	0	0	0	1
	0	0	−2	0	0	0	−6	1	62

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_4	0	0	2	1	5/2	0	0	0	6
x_1	1	0	5	0	−3	0	−2	0	−4
x_6	0	0	−2	0	4	1	−4	0	−2
x_2	0	1	0	0	3/2	0	0	0	1
	0	0	3	0	2	0	5	1	−80

(a) [2pts.] For the first tableau, what basic feasible solution does the tableau represent?

Solution: We see that it represents the basic feasible solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 5 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(b) [3pts.] For the first tableau, what is the correct entering variable if Bland's Rule for the simplex method is used, and in that case what is the departing variable?

Solution: Under Bland's Rule we choose x_3, the leftmost variable with a negative entry in the objective row, as the entering variable. The θ ratios corresponding to positive entries of the pivotal column are then $\{5/2, 3/5, 1/3\}$ for x_4, x_1, x_6 in that order; the smallest is $1/3$, so x_6 is the departing variable.

(c) [2pts.] For the first tableau, what is the correct entering variable if the standard rule for the simplex method is used, and in that case what is the departing variable?

Solution: Under the standard rule we choose x_7, the variable with the most negative entry in the objective row, as the entering variable. Then since there are no positive entries in the pivotal column, there is no valid choice of departing variable. We conclude that the linear programming problem has no finite optimal solution, that is, no maximum.

(d) [3pts.] For the second tableau, suppose I want to do a dual pivot toward making the tableau feasible. What are the correct entering and departing variables?

Solution: We let the departing variable be the variable with the most negative entry among the values of the basic variables; this is x_1. Then for the negative entries of the pivotal row, we consider the ratio of the value below that entry in the objective row to the entry in the pivotal row, which in this case gives $-2/3$ for x_5 and $-5/2$ for x_7. Since $-2/3$ is the larger of the two (or if you prefer it has the smaller absolute value) we pick the entering variable to be x_5.