Math 354, Section 04
Linear Optimization

Midterm 1

Instructions: You have 80 minutes to complete the exam. There are four questions, worth a total of 40 points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. For each of the following sets of vectors, determine whether it is linearly independent and find the dimension of its span.

   (a) [3pts.]

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 7 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

Solution: We claim this set is linearly independent (and thus that the dimension of its span is three). For suppose that we have a linear combination

$$a \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 4 \\ 3 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 1 \\ 7 \end{bmatrix} = \mathbf{0}$$

From the top row, $2a - b = 0$, or $b = 2a$. From the second row, $a + 2c = 0$, or $a = -2c$, and thus $b = -4c$. From the third row, $a - 4b + c = 0$, or after substitution $-2c + 16c + c = 0$, so $15c = 0$. Hence $c = 0$, implying that $a = 0$ and $b = 0$. So no nontrivial combination of these three vectors sums to zero, and the set is linearly independent.

   (b) [3pts.]

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$$

Solution: Labelling the vectors from left to right by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$, we observe that $\mathbf{v_2} = 2\mathbf{v_3} - \mathbf{v_1}$. Hence the set is linearly dependent. Removing $\mathbf{v_2}$ we see that the set $\{\mathbf{v_1}, \mathbf{v_3}\}$, which has the same span as the original vector, is linearly independent since neither vector is a multiple of the other. Ergo this set is a basis for the span of the original set, and the span has dimension two and is a plane in $\mathbb{R}^3$.

   (c) [4pts.]

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^5$$

Solution: Label the vectors from left to right by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}$. We notice that $\mathbf{v_4} = \mathbf{v_1} + \mathbf{v_2} + \mathbf{v_3}$, so the set is linearly dependent. We also notice that $\mathbf{v_5} = \mathbf{v_1} + \mathbf{v_3}$. So, we can drop $\mathbf{v_4}$ and $\mathbf{v_5}$ from the set and have a set of vectors with the same span. Consider this set

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

We claim this set is linearly independent. For suppose we have

$$a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0}$$

Then from the bottom row $a = 0$, and from the fourth row $a + b = 0$ so also $b = 0$, and from the third row $a + b + c = 0$ so $c = 0$ as well. Hence these three vectors are a basis for a span of the original set, and the set is three-dimensional.

2. Consider the following linear programming problem in canonical form: maximize

$$z = x_1 - x_2 + x_3$$

subject to the constraints

$$\begin{cases} 2x_1 + x_2 + x_3 + 7x_4 = 7 \\ 2x_2 - 2x_4 = 4 \\ x_1 + 4x_2 + x_3 = 12 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

(a) [7pts.] Find all of the basic solutions to the problem. Which of them are basic feasible?

Solution: In matrix notation we are interested in the equation $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 1 & 7 \\ 0 & 2 & 0 & -2 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ 12 \end{bmatrix}$$

Label the columns of A by A_1, A_2, A_3, A_4 from left to right. Since these are vectors in $\mathbb{R}^3$, we see that a basic solution $\mathbf{x}$ can have at most three variables nonzero. So there are in principle four cases to check. We however observe that $A_4 = 4A_1 - A_2$, so $\{A_1, A_2, A_4\}$ is linearly dependent and we need not check the case $x_3 = 0$. This leaves three possible cases that give basic solutions. If we let $x_4 = 0$, so that we are trying to solve $x_1A_1 + x_2A_2 + x_3A_3 = \mathbf{0}$, then we get the basic feasible solution

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

If we let $x_2 = 0$, similarly, we get the basic (but not basic feasible) solution

$$\mathbf{x} = \begin{bmatrix} 9 \\ 0 \\ 3 \\ -2 \end{bmatrix}$$

And finally if we let $x_1 = 0$, we get the basic feasible solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 9/4 \\ 3 \\ 1/4 \end{bmatrix}.$$

(b) [3pts.] Assuming that the set of feasible solutions is bounded, what is the maximum value of z subject to the constraints?

Solution: The maximum must occur at one of the two basic feasible solutions. Evaluating, we see that the maximum value is $z = 2$, which occurs at

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}.$$

3. For each of the following, draw an example of a set with the described properties. Your answers can be subsets of $\mathbb{R}$, of $\mathbb{R}^2$, or of $\mathbb{R}^3$ as you find appropriate.

   (a) [2pts.] A convex set with five extreme points.

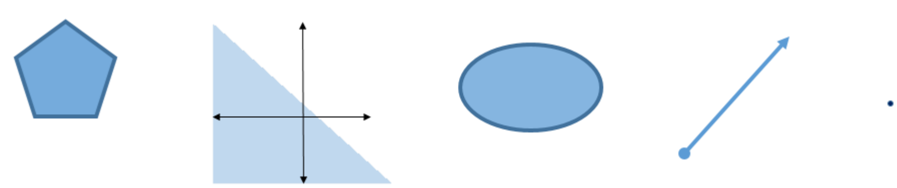
   (b) [2pts.] A convex set with no extreme points.

   (c) [2pts.] A convex set with infinitely many extreme points.

   (d) [2pts.] An unbounded convex set with exactly one extreme point.

   (e) [2pts.] A bounded convex set with exactly one extreme point.

Solution: Some possible examples are shown below, in order from left to right.



The pentagon is convex and has five extreme points, the corners. Half-spaces have no extreme points. The ellipse has infinitely many extreme points, every point on the boundary. An infinite ray has exactly one extreme point and is unbounded. A set consisting of a single point is bounded, convex, and has only one extreme point.

4. A bakery makes two kinds of doughnuts: glazed and dipped in powdered sugar. It makes a profit of 7 cents on each glazed doughnut sold and a profit of 5 cents on each powdered sugar doughnut sold. There are three main operations in doughnut making: baking (necessary for both kinds of doughnuts), glazing (necessary for the glazed doughnuts), and dipping (necessary for the powdered sugar doughnuts). The bakery's kitchen has the capacity to bake at most 1400 doughnuts, glaze at most 1000 doughnuts, and dip at most 1200 doughnuts every day. Because glazed doughnuts are very popular and important to the reputation of the store, the manager of the store has said that at least 600 should be made every day. What combination of doughnut types should the store make to maximize its profit?

   (a) [2pts.] Put this linear programming problem into a set of equations in standard form.

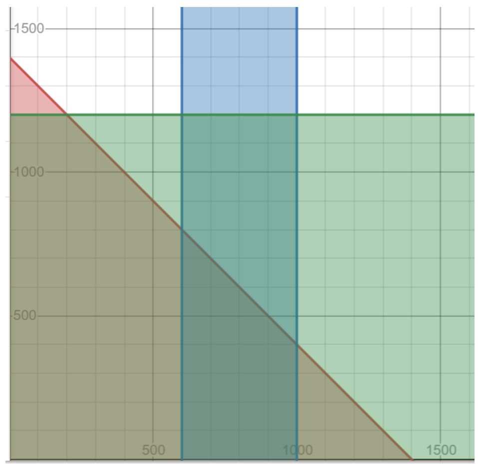
Solution: Let x_1 be the number of glazed doughnuts made and x_2 be the number of dipped doughnuts made. We wish to maximize $z = 7x_1 + 5x_2$ subject

to the constraints

$$\begin{cases} x_1 + x_2 \leq 1400 \\ x_1 \leq 1000 \\ x_2 \leq 1200 \\ -x_1 \leq -600 \\ x_1, x_2 \geq 0 \end{cases}$$

(b) [2pts.] Sketch the region of feasible solutions to your standard form model.

Solution: The region of feasible solutions is the triply-shaded quadrilateral in the graph below.



Citation: Graph from Desmos.

(c) [2pts.] Transform your equations from part (a) into canonical form.

Solution: We add four slack variables and see that we wish to maximize $z = 7x_1 + 5x_2$ subject to the constraints

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1400 \\ 1000 \\ 1200 \\ -600 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \geq \mathbf{0}.$$

(d) [2pts.] Use your sketch to identify all of the basic feasible solutions to this linear programming problem. [Don't try to find all the basic solutions, there are quite a

few.]

Solution: The basic feasible solutions correspond to the extreme points of the region of feasible solutions on the graph. We can see these occur at $(600, 0), (1000, 0)$, $(600, 800)$ and $(1000, 400)$. As basic feasible solutions of the canonical form problem these are

$$\begin{bmatrix} 600 \\ 0 \\ 800 \\ 400 \\ 1200 \\ 0 \end{bmatrix}, \begin{bmatrix} 1000 \\ 0 \\ 400 \\ 0 \\ 1200 \\ 400 \end{bmatrix}, \begin{bmatrix} 600 \\ 800 \\ 0 \\ 400 \\ 400 \\ 0 \end{bmatrix}, \begin{bmatrix} 1000 \\ 400 \\ 0 \\ 0 \\ 800 \\ 400 \end{bmatrix}.$$

(e) [2pts.] How many doughnuts of each kind should be made to maximize the bakery's profit?

Solution: The max profit must occur at one of the basic feasible solutions, or equivalently one of the extreme points of the region of feasible solutions. Evaluating at the four basic feasible solutions we see that the profit is highest at $90 when the bakery makes 1000 glazed doughnuts and 400 dipped doughnuts.