Math 354, Section 04
Linear Optimization

Sample Midterm 1

Instructions: You have 80 minutes to complete the midterm. There are four questions, worth a total of 40 points. Partial credit will be given for progress toward correct solutions where relevant. You may not use any books, notes, calculators, or other electronic devices.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. For each of the following sets of vectors, determine whether it is linearly independent and find the dimension of its span.

   (a) [3pts.]

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -12 \\ 5 \\ 4 \\ 18 \end{bmatrix} \right\} \subseteq \mathbb{R}^4$$

Solution: Labelling the vectors $\mathbf{v_1}$, $\mathbf{v_2}$, $\mathbf{v_3}$ from left to right, we observe that $\mathbf{v_3} = 5\mathbf{v_1} - 3\mathbf{v_2}$. So the set is linearly dependent. If we drop $\mathbf{v_3}$ it is clear that $\{\mathbf{v_1}, \mathbf{v_2}\}$ is linearly independent, because neither vector is a multiple of the other; we conclude this set, which has the same span as the original, is a basis for the span. Hence, the span has dimension two and is a plane in $\mathbb{R}^4$.

   (b) [3pts.]

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -9 \\ -4 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$$

Solution: Since this is a set of four vectors in $\mathbb{R}^3$, it is of course linearly dependent. Labelling the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ in order we notice that $\mathbf{v_3} = \mathbf{v_1} + \mathbf{v_2}$ and $\mathbf{v_4} = \mathbf{v_2} - 7\mathbf{v_1}$. So $\{\mathbf{v_1}, \mathbf{v_2}\}$ has the same span as the original set; it is also linearly independent since neither vector is a multiple of the other, and so we conclude the span has dimension two and is a plane in $\mathbb{R}^3$.

   (c) [4pts.]

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^5$$

Solution: We claim this set is linearly independent (and therefore that the span of the image has dimension 4). For, suppose we have that

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 3 \\ 1 \\ 4 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

The bottom line gives $d = -2a$. The line above it gives $b = -a$. The top line gives $a + 3b + c = 0$, so subbing in $b = -a$ we get $-2a + c = 0$ or $c = 2a$.

Then the second-from-the-top line gives $b + 2c + 3d = 0$, or $-a + 4a + 3d = 0$, so $d = -a$. Finally the middle row gives $a + 4b + c + 3d = 0$, which after substitutions becomes $a + 4(-a) + 2a - 3a = 0$, or $-4a = 0$. So $a = 0$ and so is every other variable, and no nontrivial linear combination of our four vectors sums to $\mathbf{0}$.

2. Consider the following linear programming problem in canonical form: maximize $z = 3x_2 - x_5$ subject to the constraints

$$\begin{cases} x_1 + 2x_2 = 6 \\ 2x_2 + x_3 + 3x_5 = 7 \\ 2x_3 + x_4 - x_5 = 0 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{cases}$$

(a) [7pts.] Find all of the basic solutions to the problem. Which of them are basic feasible?

Solution: In matrix notation, we are interested in the equation $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 3 \\ 0 & 0 & 2 & 1 & -1 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 7 \\ 0 \end{bmatrix}.$$

Label the columns of $\mathbf{A}$ from left to right as A_1, A_2, A_3, A_4, A_5 as usual. A basic solution can be nonzero in at most three entries, which corresponds to being able to write $\mathbf{b}$ as a linear combination of at most three columns. There are in principle ten different ways to choose three columns from a set of five. However, we quickly notice that $\frac{7}{2}A_2 - A_1 = \mathbf{b}$; by uniqueness of solutions to linearly independent sets of equations, we conclude that the basic solution we obtain by looking at any set of three linearly independent columns that includes A_1 and A_2 must be

$$\mathbf{x} = \begin{bmatrix} -1 \\ 7/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

We have seven cases left. But we notice that $3A_3 - 7A_4 = A_5$, so $\{A_3, A_4, A_5\}$ is linearly dependent. Six cases remain.

In the case $x_2 = x_5 = 0$, such that we are trying to solve $x_1A_1+x_3A_3+x_4A_4 = \mathbf{b}$, we get the basic solution

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 7 \\ -14 \\ 0 \end{bmatrix}.$$

Whereas if we consider the case $x_2 = x_4 = 0$, such that we are trying to solve $x_1A_1 + x_3A_3 + x_5A_5 = \mathbf{b}$ we get the basic feasible solution

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

Whereas if we consider the case $x_2 = x_3 = 0$, such that we are trying to solve $x_1A_1 + x_4A_4 + x_5A_5 = \mathbf{b}$ we get the basic feasible solution

$$\mathbf{x} = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7/3 \\ 7/3 \end{bmatrix}.$$

Next we consider the case $x_1 = x_5 = 0$, so that we are trying to solve $x_2A_2 + x_3A_3 + x_4A_4 = \mathbf{b}$. We get the basic solution

$$\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ -2 \\ 0 \end{bmatrix}.$$

Next, $x_1 = x_4 = 0$, so that we are trying to solve $x_2A_2 + x_3A_3 + x_5A_5 = \mathbf{b}$, resulting in the basic feasible solution of

$$\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 1/7 \\ 0 \\ 2/7 \end{bmatrix}.$$

And finally, $x_1 = x_3 = 0$, so that we are solving $x_2A_2 + x_4A_4 + x_5A_5 = \mathbf{b}$,

resulting in the basic feasible solution of

$$\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Remark: This problem is a bit more algebraically complicated than the roughly corresponding one on the actual exam.

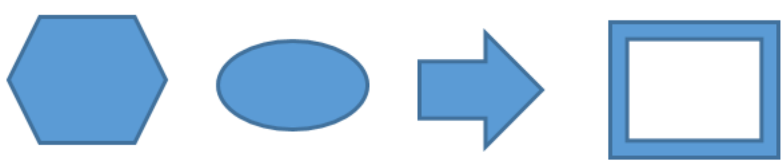
(b) [3pts.] Assuming that the set of feasible solutions is bounded, what is the maximum value of z subject to the constraints?

Solution: We have four basic feasible solutions; if the region is bounded then the there must be an optimal value and it must occur at at least one of them. Evaluating, we conclude that the maximum is $\frac{61}{7}$ and occurs at

$$\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 1/7 \\ 0 \\ 2/7 \end{bmatrix}.$$

3. (a) [6pts.] For each of the following subsets of the plane, decide whether it is convex and whether it is bounded. If if is convex, say what the extreme points are. No need to justify your answers.

- Each of the four subsets of the plane shown below.

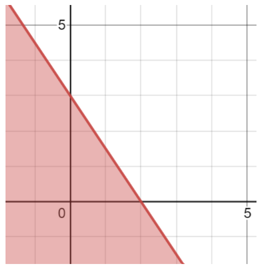


- The closed half-space $H = \{\mathbf{x} : 3x_1 + 2x_2 \leq 6\}$.
- The set of feasible solutions to

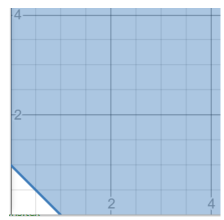
$$\begin{cases} -x_1 + -x_2 \leq -1 \\ x_1, x_2 \geq 0 \end{cases}$$

Solution: All four of the shapes on the page are bounded. The two on the left side are convex and the two on the right side are not: in both cases on the right side, you can draw a line segment between two points in the blue set that crosses the white page. The extreme points of the hexagon are its six corners; the extreme points of the ellipse are all of the points on its boundary, of which there are infinitely many.

The half space H is convex, unbounded and has no extreme points; every single point in the half-space is in the interior of some line segment.



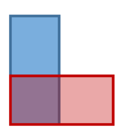
The set of feasible solutions to the equations above is convex, unbounded, and has two extreme points, namely $(1, 0)$ and $(0, 1)$.



Citation: Images from Desmos.

(b) [4pts.] In class we showed that the intersection of two convex subsets A and B in $\mathbb{R}^n$ is necessarily convex. (Recall that the intersection of A and B is the set of points that appear in both A and B, which visually looks like their overlap.) The *union* of two sets A and B is the set of all points appearing in either A or B. If A and B are convex, is their union necessarily convex? Support your answer.

Solution: It is *not* necessarily true that the union of two convex sets is convex, as we may observe from the counterexample in the image; the blue rectangle and the red rectangle are each convex, but their union is not.



4. A snack company makes chili-flavored and pizza-flavored potato chips. The chips must go through three processes: frying, flavoring, and packing. A kilogram of chili-flavored

chips takes 3 minutes to fry, 3 minutes to flavor, and 3 minutes to pack. A kilogram of pizza-flavored chips takes 3 minutes to fry, 4 minutes to flavor, and 2 minutes to pack. The net profit on a kilogram of chili-flavored chips is 10 cents and the net profit on a kilogram of pizza-flavored chips is 12 cents. The fryer is available for 4 hours every day, the flavorer is available for 4 hours every day, and the packer is available for 3 hours every day.

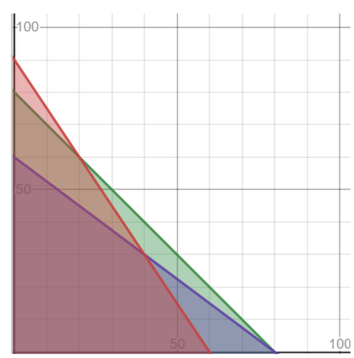
(a) [2pts.] Put this linear programming problem into a set of equations in standard form.

Solution: Let x_1 be the number of kilograms of chili-flavored chips produced and x_2 be the number of kilograms of pizza-flavored chips produced. We wish to maximize $z = 10x_1 + 12x_2$ subject to the constraints

$$\begin{cases} 3x_1 + 3x_2 \leq 240 \\ 3x_1 + 4x_2 \leq 240 \\ 3x_2 + 2x_2 \leq 180 \\ x_1, x_2 \geq 0 \end{cases}$$

(b) [2pts.] Sketch the region of feasible solutions to your standard form model.

Solution: The region of feasible solutions is the triply-shaped quadrilateral in the figure below.



(c) [2pts.] Transform your equations from part (a) into canonical form.

Solution: We add three slack variables u_1, u_2, u_3 and then see that we wish to maximize $z = 10x_1 + 12x_2$ subject to the constraints

$$\begin{bmatrix} 3 & 3 & 1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 240 \\ 240 \\ 180 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \geq \mathbf{0}$$

(d) [2pts.] Use your sketch to identify all of the basic feasible solutions to this linear programming problem. [Don't try to find all the basic solutions, there are quite a few.]

Solution: The basic feasible solutions are the extreme points of the region of feasible solutions. We can see that in the standard form picture these occur at the points $(0, 0)$, $(0, 60)$, $(60, 0)$, and $(40, 30)$. These correspond to the basic feasible solutions

$$\begin{bmatrix} 0 \\ 0 \\ 240 \\ 240 \\ 180 \end{bmatrix}, \begin{bmatrix} 60 \\ 0 \\ 60 \\ 60 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 60 \\ 60 \\ 0 \\ 60 \end{bmatrix}, \begin{bmatrix} 40 \\ 30 \\ 30 \\ 0 \\ 0 \end{bmatrix}$$

(e) [2pts.] How many chips of each kind should be made to maximize the bakery's profit?

Solution: Since the region is bounded, it suffices to check its extreme points (equivalently, basic feasible solutions). We evaluate z at the four points above to find that z takes its largest value when 40 kilos of chili-flavored chips and 30 kilos of pizza-flavored chips are made, with a total profit of $7.60.