Homework 9 Solutions

April 1, 2022

Problem 3

We recall that the primal problem was to maximize $z = .5x_1 + .8x_2 + 1.2x_3$ subject to the constraints

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 \leq 420 \\ 4x_1 + 6x_2 + 10x_3 \leq 600 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

The dual problem is to minimize $z' = 420w_1 + 600w_2$ subject to the constraints

$$\begin{cases} 2w_1 + 4w_2 \geq .5 \\ 2w_1 + 6w_2 \geq .8 \\ 3w_1 + 10w_2 \geq 1.2 \\ w_1, w_2 \geq 0 \end{cases}$$

We further recall that when we solved the primal problem using the simplex method on Homework 7 the final tableau was

	x_1	x_2	x_3	u_1	u_2	z	
u_1	2/3	0	$-1/3$	1	$-1/3$	0	220
x_2	2/3	1	5/3	0	1/6	0	100
	1/30	0	2/15	0	2/15	1	80

Since u_1 and u_2 were the initial basic variables in this computation, we conclude that the optimal solution to the dual problem occurs at

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/15 \end{bmatrix}$$

which gives the optimal value of $z' = 80$, the same as the primal problem. Since the first constraint of the primal problem represented the amount of time the sewing process is available and the second constraint represented the amount of time the gluing process is available, both in minutes, we see that at the optimal solution the marginal value of a minute of sewing process availability to us is zero and the marginal value of a minute of gluing process availability to us is 2/15 dollars.

Problem 4

We recall that the primal problem was to maximize $z = 10x_1 + 12x_2$ subject to the constraints

$$\begin{cases} 3x_1 + 3x_2 \leq 240 \\ 3x_1 + 4x_2 \leq 240 \\ 3x_1 + 2x_2 \leq 180 \\ x_1, x_2 \geq 0 \end{cases}$$

The dual problem is therefore to minimize $z = 240w_1 + 240w_2 + 180w_3$ subject to the constraints

$$\begin{cases} 3w_1 + 3w_2 + 3w_3 \geq 10 \\ 3w_1 + 4w_2 + 2w_3 \geq 12 \\ w_1, w_2, w_3 \geq 0 \end{cases}$$

We further recall that when we solved the primal problem using the simplex method on Homework 7 the final tableau was

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	30
x_2	0	1	0	$1/2$	$-1/2$	0	30
x_1	1	0	0	$-1/3$	$2/3$	0	40
	0	0	0	$8/3$	$2/3$	1	760

Since u_1, u_2, u_3 were the initial basic variables of this computation, we conclude that the optimal solution to the dual problem occurs at

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8/3 \\ 2/3 \end{bmatrix}.$$

We recall that the constraints in the primal problem corresponded to the availability of the fryer, the availability of the flavorer, and the availability of the packer in that order, given in minutes. So we conclude that at the optimal solution the marginal value of a minute of availability of the fryer to us is 0, the marginal value of a minute of availability of the flavorer to us is 8/3 cents, and the marginal value of a minute of availability of the packer to us is 2/3 cents.

Problem 5

We recall that the original problem was to maximize $z = x_1 + 3x_3$ subject to the constraints

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 4 \\ x_1 + 3x_2 + 2x_3 = 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

The dual problem is to minimize $z' = 4w_1 + 5w_2$ subject to

$$\begin{cases} w_1 + w_2 \geq 1 \\ 2w_1 + 3w_2 \geq 0 \\ 4w_1 + 2w_2 \geq 3 \end{cases}$$

where w_1 and w_2 are not constrained to be nonnegative. We recall that the conclusion of Phase I for the primal problem left us with the tableau

	x_1	x_2	x_3	y_1	y_2	z	
x_3	1/8	0	1	3/8	$-1/4$	0	1/4
x_2	1/4	1	0	$-1/4$	1/2	0	3/2
	0	0	0	1	1	1	0

We go through Phase II again, this time retaining the columns corresponding to the artificial variables. First we replace the objective row with the original objective function, retaining the columns associated to the artificial variables.

	x_1	x_2	x_3	y_1	y_2	z	
x_3	1/8	0	1	3/8	$-1/4$	0	1/4
x_2	1/4	1	0	$-1/4$	1/2	0	3/2
	-1	0	-3	0	0	1	0

Then we clear out the objective row, as previously.

	x_1	x_2	x_3	y_1	y_2	z	
x_3	1/8	0	1	3/8	$-1/4$	0	1/4
x_2	1/4	1	0	$-1/4$	1/2	0	3/2
	$-5/8$	0	0	9/8	$-3/4$	1	3/4

We pivot and obtain

	x_1	x_2	x_3	y_1	y_2	z	
x_1	1	0	8	3	-2	0	2
x_2	0	1	-2	-1	1	0	1
	0	0	5	3	-2	1	2

We see that the optimal solution to the dual problem occurs at

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

and of course the optimal value is also 2.

Problem 6

The dual to this problem is to minimize $z' = 3w_1 + 7w_2$ subject to the constraints

$$\begin{cases} w_1 + w_2 \geq 2 \\ -5w_1 \geq 3 \\ w_1 + w_2 \geq 1 \\ w_1, w_2 \geq 0 \end{cases}$$

This clearly has no feasible solutions, since $-5w_1 \geq 3$ can be rearranged to $w_1 \leq -3/5$. Since the primal problem does have feasible solutions, such as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we conclude that the primal problem has no finite optimal value.

Problem 7

Consider the problem maximize $z = x_1 + x_2$ subject to

$$\begin{cases} x_1 - x_2 \leq 1 \\ -x_1 + x_2 \leq 1 \\ x_1, x_2 \geq 0 \end{cases}$$

which has no feasible solutions and has dual problem minimize $z = w_1 + w_2$ subject to

$$\begin{cases} w_1 - w_2 \geq 1 \\ -w_2 - w_1 \geq 1 \\ w_1, w_2 \geq 0 \end{cases}$$

which also has no feasible solutions.