Homework 8 Solutions

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Problem 3

We want to maximize to maximize $z = x_1 + 3x_3$ subject to the constraints

$$\begin{cases} x_1 + 2x_2 + 4x_3 = 4 \\ x_1 + 3x_2 + 2x_3 = 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

We note this is already in canonical form with the right-hand side positive. For Phase I, we add two artificial variables

$$\begin{cases} x_1 + 2x_2 + 4x_3 + y_1 = 4 \\ x_1 + 3x_2 + 2x_3 + y_2 = 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

and first solve the auxiliary problem of mazimizing $z = -y_1 - y_2$ subject to the constraints above. This gives us the tableau

	x_1	x_2	x_3	y_1	y_2	z	
y_1	1	2	4	1	0	0	4
y_2	1	3	2	0	1	0	5
	0	0	0	1	1	1	0

We then clear out the entries in the objective row under the basic variables to obtain an initial tableau for the auxiliary problem, shown below.

	x_1	x_2	x_3	y_1	y_2	z	
y_1	1	2	4	1	0	0	4
y_2	1	3	2	0	1	0	5
	-2	-5	-6	0	0	1	-9

The entering variable is x_3 since -6 is the most negative entry in the objective row. The θ-ratios from top to bottom are $\{4/4, 5/2\}$, of which the first is smaller, so we choose y_1 for the departing variable. We then perform row operations to obtain a new tableau:

	x_1	x_2	x_3	y_1	y_2	z	
x_3	$1/4$	$1/2$	1	$1/4$	0	0	1
y_2	$1/2$	2	0	$-1/2$	1	0	3
	$-1/2$	-2	0	$3/2$	0	1	-3

The new entering variable is x_2 since -2 is the most negative entry in the objective row. The θ-ratios from top to bottom are $\{1/(1/2), 3/2\}$, of which $3/2$ is the smaller, so we choose y_2 for the departing variable. We perform row operations to obtain a new tableau:

	x_1	x_2	x_3	y_1	y_2	z	
x_3	1/8	0	1	3/8	$-1/4$	0	1/4
x_2	1/4	1	0	$-1/4$	1/2	0	3/2
	0	0	0	1	1	1	0

We have now finished solving the auxiliary problem and found a basic feasible solution to the original problem, namely

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2 \\ 1/4 \end{bmatrix}.$$

Hence Phase I is done. We now solve the original problem for Phase II. We start by dropping the columns for the artificial variables and inserting the original objective function

	x_1	x_2	x_3	z	
x_3	1/8	0	1	0	1/4
x_2	1/4	1	0	0	3/2
	-1	0	-3	1	0

We then clear out the entries below the basic variables to obtain a new initial tableau.

	x_1	x_2	x_3	z	
x_3	1/8	0	1	0	1/4
x_2	1/4	1	0	0	3/2
	$-5/8$	0	0	1	3/4

The new entering variable is x_1 since it has the only negative entry on the objective row. The θ-ratios from top to bottom are $\{(1/4)/(1/8), (3/2)/(1/4)\} = \{2, 6\}$ so we let x_3 be the departing variable. We then pivot and obtain

	x_1	x_2	x_3	z	
x_1	1	0	8	0	2
x_2	0	1	-2	0	1
	0	0	5	1	2

Since there are no further positive entries in the objective row, we are done! The optimal value of z is 2 and the optimal solution is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Problem 4

We are interested in maximizing $z = x_1 + 2x_2 + 7x_3 - x_4$ subject to the constraints

$$\begin{cases} 3x_1 + x_2 - x_4 = 2 \\ 2x_1 + 4x_2 + 7x_3 \geq 3 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$

First, we put the constraints in canonical form by adding a single slack variable as follows.

$$\begin{cases} 3x_1 + x_2 - x_4 = 2 \\ 2x_1 + 4x_2 + 7x_3 - u_1 = 3 \\ x_1, x_2, x_3, x_4, u_1 \geq 0 \end{cases}$$

We see that we need two artificial variables in order to end up with a set of basic variables each appearing in one equation with coefficient 1. We add them:

$$\begin{cases} 3x_1 + x_2 - x_4 + y_1 = 2 \\ 2x_1 + 4x_2 + 7x_3 - u_1 + y_2 = 3 \\ x_1, x_2, x_3, x_4, u_1, y_1, y_2 \geq 0 \end{cases}$$

Now we let M be some large positive number and adjust our objective function to be $z = x_1 + 2x_2 + 7x_3 - x_4 - My_1 - My_2$. This means that we begin with

	x_1	x_2	x_3	x_4	u_1	y_1	y_2	z	
y_1	3	1	0	-1	0	1	0	0	2
y_2	2	4	7	0	-1	0	1	0	3
	-1	-2	-7	1	0	M	M	1	0

Our first step is to clear out the M entries under the basic variables so that we have an initial tableau:

	x_1	x_2	x_3	x_4	u_1	y_1	y_2	z	
y_1	3	1	0	-1	0	1	0	0	2
y_2	2	4	7	0	-1	0	1	0	3
	$-1-5M$	$-2-5M$	$-7-7M$	$1+M$	0	0	0	1	$-5M$

Bearing in mind that M is large and positive, we observe that the most negative entry in the objective row is $-7-7M$. Thus the entering variable is x_3. There is a single positive entry in the pivotal column, so the departing variable is y_2. We pivot and obtain the following tableau.

	x_1	x_2	x_3	x_4	u_1	y_1	y_2	z	
y_1	3	1	0	-1	0	1	0	0	2
x_3	$2/7$	$4/7$	1	0	$-1/7$	0	$1/7$	0	$3/7$
	$1-3M$	$2-M$	0	$1+M$	-1	0	$1+M$	1	$3-2M$

Now the most negative entry in the objective row is $1-3M$, so we let x_1 be the entering variable. We see the θ-ratios from top to bottom are $\{2/3, (3/7)/(2/7)\} = \{2/3, 3/2\}$, so the departing variable is y_1. We pivot accordingly and obtain the following tableau.

	x_1	x_2	x_3	x_4	u_1	y_1	y_2	z	
x_1	1	$1/3$	0	$-1/3$	0	$1/3$	0	0	$2/3$
x_3	0	$10/21$	1	$2/21$	$-1/7$	$-2/21$	$1/7$	0	$5/21$
	0	$5/3$	0	$4/3$	-1	$3M-1$	$1+M$	1	$11/3$

We only have a single negative entry in the objective row, and there are no positive entries above it in the pivotal column. So this function has no maximum with respect to these constraints!

Problem 5

We recall that we are interested in maximizing

$$z = .09x_1 + .04x_2 + .05x_3$$

and the constraints are

$$\begin{cases} x_1 + x_2 + x_3 \leq 200000 \\ x_1 + x_2 - x_3 \leq 0 \\ x_1 \leq 40000 \\ x_3 \geq 70000 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

For the sake of our sanity we will instead maximize $z = 90x_1 + 40x_2 + 50x_3$ subject to the constraints

$$\begin{cases} x_1 + x_2 + x_3 \leq 200 \\ x_1 + x_2 - x_3 \leq 0 \\ x_1 \leq 40 \\ x_3 \geq 70 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

since this problem is equivalent. (All we've done is make the variables x_1, x_2, and x_3 be in units of thousands of dollars rather than dollars and leave z in dollars.) We add slack variables to put this problem into canonical form:

$$\begin{cases} x_1 + x_2 + x_3 + u_1 = 200 \\ x_1 + x_2 - x_3 + u_2 = 0 \\ x_1 + u_3 = 40 \\ x_3 - u_4 = 70 \\ x_1, x_2, x_3, u_1, u_2, u_3, u_4 \geq 0 \end{cases}$$

We almost have a set of basic variables that we could use for an initial tableau – each of u_1, u_2, u_3 appears in a single equation with coefficient 1. We just need to add a single artificial variable y_1 to the last equation.

$$\begin{cases} x_1 + x_2 + x_3 + u_1 = 200 \\ x_1 + x_2 - x_3 + u_2 = 0 \\ x_1 + u_3 = 40 \\ x_3 - u_4 + y_1 = 70 \\ x_1, x_2, x_3, u_1, u_2, u_3, u_4 \geq 0 \end{cases}$$

Now we can get going on Phase I by maximizing $z = -y_1$ with respect to these constraints. We start with the following.

	x_1	x_2	x_3	u_1	u_2	u_3	u_4	y_1	z	
u_1	1	1	1	1	0	0	0	0	0	200
u_2	1	1	-1	0	1	0	0	0	0	0
u_3	1	0	0	0	0	1	0	0	0	40
y_1	0	0	1	0	0	0	-1	1	0	70
	0	0	0	0	0	0	0	1	1	0

We clear out the nonzero entry under the basic variable y_1 to get an initial tableau

	x_1	x_2	x_3	u_1	u_2	u_3	u_4	y_1	z	
u_1	1	1	1	1	0	0	0	0	0	200
u_2	1	1	-1	0	1	0	0	0	0	0
u_3	1	0	0	0	0	1	0	0	0	40
y_1	0	0	1	0	0	0	-1	1	0	70
	0	0	-1	0	0	0	1	0	1	-70

We see our entering variable is x_3 and the departing variable is y_1 since $70/1$ is the smallest positive θ-ratio. We pivot and obtain a new tableau

	x_1	x_2	x_3	u_1	u_2	u_3	u_4	y_1	z	
u_1	1	1	0	1	0	0	1	-1	0	130
u_2	1	1	0	0	1	0	-1	1	0	70
u_3	1	0	0	0	0	1	0	0	0	40
x_3	0	0	1	0	0	0	-1	1	0	70
	0	0	0	0	0	0	0	1	1	0

As there are no more negative entries in the objective row, Phase I is now complete; we have a basic feasible solution to the original problem. Now for Phase II. We drop the column corresponding to the artificial variable and reintroduce the original objective function, resulting in

	x_1	x_2	x_3	u_1	u_2	u_3	u_4	z	
u_1	1	1	0	1	0	0	1	0	130
u_2	1	1	0	0	1	0	-1	0	70
u_3	1	0	0	0	0	1	0	0	40
x_3	0	0	1	0	0	0	-1	0	70
	-90	-40	-50	0	0	0	0	1	0

We clear out the entry under the basic variable x_3 to obtain a new initial tableau

	x_1	x_2	x_3	u_1	u_2	u_3	u_4	z	
u_1	1	1	0	1	0	0	1	0	130
u_2	1	1	0	0	1	0	-1	0	70
u_3	1	0	0	0	0	1	0	0	40
x_3	0	0	1	0	0	0	-1	0	70
	-90	-40	0	0	0	0	-50	1	350

The entering variable is x_1 and the departing variable is then u_3. We pivot accordingly and obtain the following new tableau.

	x_1	x_2	x_3	u_1	u_2	u_3	u_4	z	
u_1	0	1	0	1	0	-1	1	0	90
u_2	0	1	0	0	1	-1	-1	0	30
x_1	1	0	0	0	0	1	0	0	40
x_3	0	0	1	0	0	0	-1	0	70
	0	-40	0	0	0	90	-50	1	710

Now u_4 enters and u_1 departs. We pivot again.

	x_1	x_2	x_3	u_1	u_2	u_3	u_4	z	
u_4	0	1	0	1	0	-1	1	0	90
u_2	0	2	0	1	1	-2	0	0	120
x_1	1	0	0	0	0	1	0	0	40
x_3	0	1	1	1	0	-1	0	0	160
	0	10	0	0	0	40	0	1	1160

There are no more negative entries in the objective row, so we are done. The maximum return on investments is \$1160 per year when \$40,000 is invested in the utilities stock x_1 and \$160,000 is invested in the bond x_3.

Problem 6

We confirm using the online calculator that the simplex algorithm using the standard rule cycles after six steps; the cycle of sets of basic variables is $\{x_4, x_5, x_6, x_7\} \to \{x_1, x_5, x_6, x_7\} \to \{x_1, x_2, x_6, x_7\} \to \{x_1, x_2, x_3, x_7\} \to \{x_2, x_3, x_4, x_7\} \to \{x_3, x_4, x_5, x_7\} \to \{x_4, x_5, x_6, x_7\}$. We see that the first place that the standard rule and Bland's rule give different instructions is on the last step. So we start with the tableau

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_4	-9	32	0	1	0	-24	0	0	0
x_5	-1	3	0	0	1	-3	0	0	0
x_3	2	-8	1	0	0	5	0	0	0
x_7	0	1	0	0	0	0	1	0	1
	-4	14	0	0	0	-9	0	1	0

Using Bland's Rule, we let the entering variable be x_1, which makes the departing variable x_3. We then pivot and obtain the tableau below.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_4	0	-4	9	1	0	$-3/2$	0	0	0
x_5	0	-1	1	0	1	$-1/2$	0	0	0
x_1	1	-4	1	0	0	$5/2$	0	0	0
x_7	0	1	0	0	0	0	1	0	1
	0	-2	0	0	0	1	0	1	0

Now we have an entering variable of x_2 and thus a departing variable of x_7. We perform one final pivot.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	z	
x_4	0	0	9	1	0	$-3/2$	4	0	4
x_5	0	0	1	0	1	$-1/2$	1	0	1
x_1	1	0	1	0	0	$5/2$	4	0	4
x_2	0	1	0	0	0	0	1	0	1
	0	0	0	0	0	1	2	1	2

We see that we are now done! The optimal value of z is 2 at the basic feasible solution

$$\begin{bmatrix} 4 \\ 1 \\ 0 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$