Homework 7 Solutions

April 1, 2022

Problem 3

We recall that the canonical form of this problem is to maximize $z = .5x_1 + .8x_2 + 1.2x_3$ subject to the constraints

$$\begin{bmatrix} 2 & 2 & 3 & 1 & 0 \\ 4 & 6 & 10 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 420 \\ 600 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ u_1 \\ u_2 \end{bmatrix} \geq 0.$$

As per last week, this becomes the initial tableau

	x_1	x_2	x_3	u_1	u_2	z	
u_1	2	2	3	1	0	0	420
u_2	4	6	10	0	1	0	600
	$-.5$	$-.8$	-1.2	0	0	1	0

which corresponds to a z-value of 0 at the initial basic feasible solution

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 420 \\ 600 \end{bmatrix}.$$

The most negative entry in the objective row of the tableau is -1.2 so the entering variable is x_3, and its column is the pivotal column. The θ-ratios are then $\{420/3, 600/10\}$. The smallest positive θ-ratio is $600/10 = 60$, so we let u_2 be the departing variable; the pivot is the entry 10 in the x_3 column and u_2 row. We perform row operations to transform the x_3 column into an identity column. First we divide the pivotal row by the coefficient on the pivot.

	x_1	x_2	x_3	u_1	u_2	z	
	2	2	3	1	0	0	420
	.4	.6	1	0	.1	0	60
	$-.5$	$-.8$	-1.2	0	0	1	0

Then we clear out the remaining entries in the pivotal column by adding multiples of the pivotal row to the other rows. We get the following new tableau

	x_1	x_2	x_3	u_1	u_2	z	
u_1	.8	.2	0	1	$-.3$	0	240
x_3	.4	.6	1	0	.1	0	60
	$-.02$	$-.08$	0	0	.12	1	72

Now we are at the basic feasible solution

$$\begin{bmatrix} 0 \\ 0 \\ 60 \\ 240 \\ 0 \end{bmatrix}$$

where $z = 72$. There are still negative entries in the objective row, so we can increase this. The most negative entry is $-.08$, so we pick x_2 for the entrering variable. The θ-ratios are then $\{240/.2, 60/.6\} = \{1200, 100\}$, so x_3 is the departing variable. The pivot is the entry .6 at the intersection of the pivotal row and pivotal column. We start by dividing the pivotal row by the coefficient on the pivot.

	x_1	x_2	x_3	u_1	u_2	z	
	.8	.2	0	1	$-3/10$	0	240
	2/3	1	5/3	0	1/6	0	100
	$-.02$	$-.08$	0	0	.12	1	72

Then we clear out the remaining entries in the pivotal column by adding multiples of the pivotal row to the other rows. We get the following new tableau

	x_1	x_2	x_3	u_1	u_2	z	
u_1	2/3	0	$-1/3$	1	$-1/3$	0	220
x_2	2/3	1	5/3	0	1/6	0	100
	1/30	0	2/15	0	2/15	1	80

corresponding to the basic feasible solution

$$\begin{bmatrix} 0 \\ 100 \\ 0 \\ 220 \\ 0 \end{bmatrix}$$

and a z-value of 80. As there are no more negative entries on the objective row, we are done. The publishers should make 100 book club edition books for \$80 profit.

Problem 4

We recall that the canonical form of this problem is to mazimize $z = 10x_1 + 12x_2$ subject to the constraints

$$\begin{bmatrix} 3 & 3 & 1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 240 \\ 240 \\ 180 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} \geq \mathbf{0}.$$

As per last week, this becomes the initial tableau shown below.

	x_1	x_2	u_1	u_2	u_3	z	
u_1	3	3	1	0	0	0	240
u_2	3	4	0	1	0	0	240
u_3	3	2	0	0	1	0	180
	-10	-12	0	0	0	1	0

corresponding to a z-value of 0 at the initial basic feasible solution

$$\begin{bmatrix} 0 \\ 0 \\ 240 \\ 240 \\ 180 \end{bmatrix}.$$

We see the most negative entry on the objective row is the -12, so we let x_2 be the entering variable. Then the θ-ratios are $\{240/3, 240/4, 180/2\}$, the smallest of which is $240/4 = 60$, so we let u_2 be the departing variable. We start by dividing the pivotal row by the coefficient on the pivot, obtaining

	x_1	x_2	u_1	u_2	u_3	z	
	3	3	1	0	0	0	240
	3/4	1	0	1/4	0	0	60
	3	2	0	0	1	0	180
	-10	-12	0	0	0	1	0

Then we clear out the remaining entries in the pivotal column by adding multiples of the pivotal row to the other rows. We get the following new tableau

	x_1	x_2	u_1	u_2	u_3	z	
u_1	3/4	0	1	$-3/4$	0	0	60
x_2	3/4	1	0	1/4	0	0	60
u_3	3/2	0	0	$-1/2$	1	0	60
	-1	0	0	3	0	1	720

corresponding to the basic feasible solution

$$\begin{bmatrix} 0 \\ 60 \\ 60 \\ 0 \\ 60 \end{bmatrix}$$

at which $z = 720$. There are is still a negative entry in the objective row, so we could still increase z. We let x_1 be the entering variable. Then the θ-ratios are $\{60/(3/4), 60/(3/4), 60/(3/2)\}$, of which the smallest is $60/(3/2) = 40$. So, u_3 will be the departing variable. We start out by dividing the pivotal row by the coefficient on the pivot, obtaining the following.

	x_1	x_2	u_1	u_2	u_3	z	
	3/4	0	1	$-3/4$	0	0	60
	3/4	1	0	1/4	0	0	60
	1	0	0	$-1/3$	2/3	0	40
	-1	0	0	3	0	1	720

Then we clear out the remaining entries in the pivotal column by adding multiples of the pivotal row to the other rows. We get the following new tableau

	x_1	x_2	u_1	u_2	u_3	z	
u_1	0	0	1	$-1/2$	$-1/2$	0	30
x_2	0	1	0	1/2	$-1/2$	0	30
x_1	1	0	0	$-1/3$	2/3	0	40
	0	0	0	8/3	2/3	1	760

which corresponds to the basic feasible solution

$$\begin{bmatrix} 40 \\ 30 \\ 30 \\ 0 \\ 0 \end{bmatrix}$$

at which $z = 760$. There are no more negative entries in the objective row, so we are done; the snack company should make 40 kilos of chili-flavored chips and 30 kilos of pizza-flavored chips, for a total profit of \$7.60.

Problem 5

We start with the problem being to maximize $z = x_1 + 3x_2 + 5x_3$ subject to the constraints

$$\begin{cases} 2x_1 - 5x_2 + x_3 \leq 3 \\ x_1 + x_3 \leq 5 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

First, we add two slack variables to put this in canonical form, turning the constraints into

$$\begin{cases} 2x_1 - 5x_2 + x_3 + u_1 = 3 \\ x_1 + x_3 + u_2 = 5 \\ x_1, x_2, x_3, u_1, u_2 \geq 0 \end{cases}$$

Now, we may set up an initial tableau

	x_1	x_2	x_3	u_1	u_2	z	
u_1	2	-5	1	1	0	0	3
u_2	1	0	1	0	1	0	5
	-1	-3	-5	0	0	1	0

coresponding to a z-value of 0 at the initial basic feasible solution

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 3 \\ 5 \end{bmatrix}.$$

The most negative entry on the objective row is the -5, so we select x_3 for the entering variable. The θ-ratios are then $\{3/1, 5/1\}$; the smaller one is $3/1 = 3$, so we choose u_1 for the entering variable. Since the coefficient on the pivot is already 1, we skip straight to clearing out the rest of the pivotal column, obtaining the following new tableau

	x_1	x_2	x_3	u_1	u_2	z	
x_3	2	-5	1	1	0	0	3
u_2	-1	5	0	-1	1	0	2
	9	-28	0	5	0	1	15

which corresponds to the basic feasible solution

$$\begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 2 \end{bmatrix}$$

with a z-value of 15. Now the only negative entry in the objective row is the -28, so we let x_2 be the new entering variable. The θ-ratios are then $\{3/-5, 2/5\}$ and the only positive one is the $2/5$, so we let u_2 be the departing variable. We divide the pivotal row by the coefficient of the pivot and then clear out the rest of the pivotal column, obtaining the new tableau

	x_1	x_2	x_3	u_1	u_2	z	
x_3	1	0	1	0	0	0	5
x_2	$-1/5$	1	0	$-1/5$	$1/5$	0	$2/5$
	$17/5$	0	0	$-3/5$	0	1	$131/5$

which corresponds to the basic feasible solution

$$\begin{bmatrix} 0 \\ 2/5 \\ 5 \\ 0 \\ 0 \end{bmatrix}.$$

with a z-value of $131/5$. Now we still have a negative entry in the objective row, so we could increase z by increasing u_1. But the θ-ratios would then be $\{5/0, (2/5)/(-1/5)\}$, none of which is positive, so there is no bound on how much we can increase u_1. Hence, z has no maximum value with respect to these constraints!

Problem 6

The tableau corresponds to the basic feasible solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 6 \\ 0 \\ 5 \\ 0 \end{bmatrix}.$$

If x_3 is the entering variable, the θ-ratios are $\{6/2, 2/5, 5/3, 1/0\}$ in order from top to bottom. The smallest positive θ-ratio is $2/5 = .4$, so the entering variable is x_1. If x_5 is the entering variable, the θ-ratios are $\{6/(5/2), 2/-3, 5/4, 1/(3/2)\}$ in order from top to bottom. The smallest positive θ-ratio is $1/(3/2) = 2/3$, so the entering variable is x_2. Finally, if x_7 is the entering variable, then the θ-ratios are $\{6/0, 2/-2, 5/-4, 0/1\}$. Since none of these are positive, none of them give a bound on how much we can increase x_7; we see that there is no departing variable and we may increase the function arbitrarily by increasing x_7. In this case there is no maximum value to the objective function.