Homework 5 Solutions

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Problem 4

We recall from Homework 1 that the problem was to maximize

$$z = .5x_1 + .8x_2 + 1.2x_3$$

subject to the constraints

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 \leq 420 \\ 4x_1 + 6x_2 + 10x_3 \leq 600 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

We change this to canonical form by adding two slack variables, obtaining

$$\begin{cases} 2x_1 + 2x_2 + 3x_3 + u_1 = 420 \\ 4x_1 + 6x_2 + 10x_3 + u_2 = 600 \\ x_1, x_2, x_3, u_1, u_2 \geq 0 \end{cases}$$

The first two lines give us the matrix equation

$$\begin{bmatrix} 2 & 2 & 3 & 1 & 0 \\ 4 & 6 & 10 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 420 \\ 600 \end{bmatrix}$$

This matrix has rank two and the basic solutions must have at most two nonzero entries. There are ten possible ways to have two variables nonzero. The basic solutions which are not feasible are

$$\begin{bmatrix} 210 \\ 0 \\ 0 \\ 0 \\ -240 \end{bmatrix}, \begin{bmatrix} 0 \\ 210 \\ 0 \\ 0 \\ -660 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 140 \\ 0 \\ -800 \end{bmatrix}, \begin{bmatrix} 330 \\ -120 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 300 \\ 0 \\ -60 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1200 \\ -660 \\ 0 \\ 0 \end{bmatrix}$$

and the basic feasible solutions are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 420 \\ 600 \end{bmatrix}, \begin{bmatrix} 150 \\ 0 \\ 0 \\ 120 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 100 \\ 0 \\ 220 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 60 \\ 240 \\ 0 \end{bmatrix}$$

Bearing in mind that the profit clearly must have a maximum value, we see the max must occur at one of the four basic feasible solutions. Evaluating z on all of them, we conclude that the maximum profit is $80 and occurs when the publisher makes 100 book club edition books.

Problem 5

We have the constraints

$$\begin{cases} x_1 + x_2 + x_3 \leq 200000 \\ x_1 + x_2 - x_3 \leq 0 \\ x_1 \leq 40000 \\ x_3 \geq 70000 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

which we may turn into canonical form by adding four slack variables

$$\begin{cases} x_1 + x_2 + x_3 + u_1 = 200,000 \\ x_1 + x_2 - x_3 + u_2 = 0 \\ x_1 + u_3 = 40,000 \\ -x_3 + u_4 = -70,000 \\ x_1, x_2, x_3, u_1, u_2, u_3, u_4 \geq 0 \end{cases}$$

In matrix notation we want to find basic solutions to

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 200,000 \\ 0 \\ 40,000 \\ -70,000 \end{bmatrix}$$

A few possible basic solutions are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 200,000 \\ 0 \\ 40,000 \\ -70,000 \end{bmatrix}, \begin{bmatrix} 40,000 \\ 30,000 \\ 70,000 \\ 60,000 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 130,000 \\ 0 \\ 70,000 \\ 0 \\ -60,000 \\ -90,000 \\ 0 \end{bmatrix}$$

The main thing to look out for is that not all sets of four columns above are linearly independent – as a check, note that if you try to find a solution with a linearly dependent set of columns, you'll get a line of solutions instead of a single basic solution.

Problem 6

We consider the matrix equation

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & 4 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \geq \mathbf{0}.$$

Label the columns A_1, A_2, A_3, A_4. Since there are three rows, the basic solutions $\mathbf{x}$ can at most three nonzero entries. We have four possibilities, which we now investigate. First we suppose that $\mathbf{x}$ has $x_4 = 0$. We notice that $\{A_1, A_2, A_3\}$ is indeed a linearly independent set. Now we are looking for solutions to

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix}$$

The solution to this is $x_1 = -1, x_2 = 3, x_2 = 3$. So we have a basic (but not basic feasible) solution

$$\mathbf{x} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 0 \end{bmatrix}.$$

Next we consider $x_3 = 0$, noticing that $\{A_1, A_2, A_4\}$ is linearly independent. We are looking for a solution to

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix}$$

The solution to this is $x_1 = 5, x_2 = -3, x_4 = 2$, so we have a basic (but not basic feasible) solution

$$\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 0 \\ 2 \end{bmatrix}.$$

On to $x_2 = 0$, noting that $\{A_1, A_3, A_4\}$ is linearly independent. We are now trying to solve

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 1 \end{bmatrix}$$

We see the solution is $x_1 = 2, x_3 = 1, x_4 = 1$, so we have a basic feasible solution

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Finally we check $x_1 = 0$, since $\{A_2, A_3, A_4\}$ is also linearly independent. We see that $x_2 = 2, x_3 = 5/3, x_4 = 1/3$. So we have a basic feasible solution of

$$\mathbf{x} = \begin{bmatrix} 0 \\ 2 \\ 5/3 \\ 1/3 \end{bmatrix}.$$

Given that $z = 2x_1 + x_3$ must take a maximum on a bounded region at one of the basic feasible solutions, we conclude the maximum value of z is 5, and occurs at the first basic feasible solution we found above.

Problem 7

Let $\mathbf{Ax} = \mathbf{b}$ be the matrix equation of the first constraint, with the columns of $\mathbf{A}$ labelled in order $A_1, \ldots, A_5$. We first notice that it is not the case that $\mathbf{Av} = \mathbf{b}$, so $\mathbf{v}$ is not actually a solution at all. The remaining three vectors all solve the system. We consider $\mathbf{w}$. For this vector the set of columns corresponding to the nonzero entries is $\{A_2, A_3, A_5\}$, which is a linearly dependent set. It is therefore not a basic solution (or a feasible solution, since some entries are negative). We now look at $\mathbf{y}$; here the set of columns corresponding to the nonzero entries of the set is $\{A_3, A_5\}$, which is linearly independent. So $\mathbf{y}$ is a basic feasible solution. Finally, we see that for $\mathbf{z}$ the set of columns corresponding to the nonzero entries is $\{A_2, A_3, A_5\}$, which is linearly dependent. So this is a feasible solution but not a basic one.