Homework 4 Solutions

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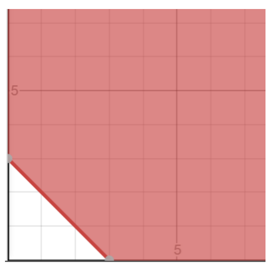
Problem 3

We may for example consider the problem of maximizing $z = x_1 + x_2$ subject to the constraints

$$\begin{cases} -x_1 - x_2 \leq -3 \\ x_1, x_2 \geq 0 \end{cases}$$

which has the unbounded region of feasible solutions shown below. We see that the function z has no maximum on this set.



Problem 4

Let S be a bounded subset of $\mathbb{R}^n$ contained in the rectangle $R = \{\mathbf{x} : -M_j \leq x_j \leq M_j\}$ for all $j = 1, \dots n$. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be given by multiplication by an $m \times n$ matrix D whose entries are denoted d_{ij}. Let $\mathbf{x}$ lie in S, so that for each $j = 1, \dots, n$ we have that $a_j \leq x_j \leq b_j$. Then notice that the ith coordinate y_i of $\mathbf{y} = f(\mathbf{x})$ is

$$y_i = d_{i1}x_1 + d_{i2}x_2 + \cdots + d_{in}x_n.$$

We notice that if $d_{ij} \geq 0$ then $-d_{ij}M_j \leq d_{ij}x_j \leq d_{ij}M_j$, whereas if $d_{ij} < 0$ then $d_{ij}M_j \leq d_{ij}x_j \leq -d_{ij}M_j$. Either way we have that

$$-|d_{ij}|M_j \leq d_{ij}x_j \leq |d_{ij}|M_j$$

and therefore that

$$-\sum_{j=1}^{n} |d_{ij}|M_j \leq y_i \leq \sum_{j=1}^{n} |d_{ij}|M_j|.$$

This implies that $\mathbf{y}$ lies in the rectangle

$$R' = \left\{ \mathbf{z} : -\sum_{j=1}^{n} |d_{ij}|M_j \leq z_i \leq \sum_{j=1}^{n} |d_{ij}|M_j \right\} \subseteq \mathbb{R}^m.$$

Since $\mathbf{y} = f(\mathbf{x})$ was the image of any element of S, we see that all elements in $f(S)$ are contained in R'. So, $f(S)$ is a bounded subset of $\mathbb{R}^m$.

Problem 5

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ on the open interval $S = (0, 1) \subset \mathbb{R}$. We see that S is bounded (for example it is contained in the rectangle $[0, 1] \subset R$, but $f(S) = (1, \infty)$ is not bounded, since it is contained in no finite rectangle.

Problem 6

(a) We claim this set is linearly dependent. For observe that

$$-2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \mathbf{0}.$$

However, we note that the set

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$$

is linearly independent, since if we have a and b solving

$$a\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \mathbf{0}$$

then we must have all of $a + 3b = 0$, $2a + 4b = 0$, and $5b = 0$ true simultaneously, to which the only solution is $a = b = 0$. Ergo these two vectors span a plane, which is also the span of the original set. Hence the span is two-dimensional.

(b) We claim this set is linearly dependent. For observe that

$$7\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 7 \\ 14 \end{bmatrix} = \mathbf{0}$$

The span of this set is the line given by all multiples of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; it is one-dimensional.
(c) We claim this set is linearly independent. For, suppose there are real numbers a, b such that

$$a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \mathbf{0}.$$

Then we must have $a + 2b = 0$, $2a + b = 0$, and $3a = 0$ all true at once. The only solution to this is $a = b = 0$. These two vectors span a two-dimensional plane in $\mathbb{R}^3$.

(d) This set is linearly independent (as any set consisting of exactly one nonzero vector is) and has the same span as part (b).

(e) We claim this set is linearly independent. For suppose we have a, b, c solving

$$a\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} + c\begin{bmatrix} 0 \\ 2 \\ -7 \end{bmatrix} = \mathbf{0}.$$

It then must be the case that all of

$$\begin{cases} a - 2b = 0 \\ 2a + 3b + 2c = 0 \\ a + 5b - 7c = 0 \end{cases}$$

The first equation gives us $a = 2b$, which turns the remaining two equations into $7b + 2c = 0$ and $7b - 7c = 0$. From the last equation, we now have $b = c$, so the middle equation becomes $9b = 0$, implying that $b = c = a = 0$. Since this is a linearly independent set of three vectors in $\mathbb{R}^3$, its span is $\mathbb{R}^3$.

Problem 7

We are looking for vectors $\mathbf{v_3}$ with the property that any linear combination $a_1\mathbf{v_1}+a_2\mathbf{v_2}+a_3\mathbf{v_3} = \mathbf{0}$ must have only the solution $a_1 = a_2 = a_3 = 0$. Suppose that instead the system has a solution with some a_i nonzero. Then we note that we cannot have $a_3 = 0$, because this would imply that there was a linear combination $a_1v_1 + a_2v_2 = 0$ with at least one of a_1 and a_2 nonzero, and this is false since by part (c) of Problem 6 the two vectors are linearly independent. So, in this case $a_3 \neq 0$, and we may write

$$\mathbf{v_3} = -\frac{a_1}{a_3}\mathbf{v_1} + -\frac{a_2}{a_3}\mathbf{v_2}$$

hence $\mathbf{v_3}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$. Conversely, suppose $\mathbf{v_3}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$, written as $\mathbf{v_3} = b_1\mathbf{v_1} + b_2\mathbf{v_2}$; then we have $b_1\mathbf{v_1} + b_2\mathbf{v_2} - \mathbf{v_3} = \mathbf{0}$, so the set is linearly dependent. So $\mathbf{v_3}$ makes the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ linearly independent exactly if it is not a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$, which is to say if it is not on the plane spanned by those two vectors. (The plane in question is $x - 2y + z = 0$, but you don't have to know this.)